The Intuitive Guide to

Fourier Analysis & Spectral Estimation

with MATLAB ®

This book will deepen your understanding of Fourier analysis making it easier to advance to more complex topics in digital signal processing and data analysis in mathematics, physics, astronomy, bio-sciences, and financial engineering. With numerous examples, detailed explanations, and plots, we make the difficult concepts clear and easy to grasp.

Fourier transform developed slowly, from the Fourier series 200 years ago to Fourier transform as implemented by the FFT today. We tell you this story, in words and equations and help you understand how each step came about.

- We start with the development of Fourier series using harmonic sinusoids to represent periodic signals in continuous and discrete-time domains.
- From here, we examine the complex exponential to represent the Fourier series basis functions.
- Next, we describe the development of the continuous-time and discrete-time Fourier transforms (CTFT, DTFT) for non-periodic signals.
- We show how the DTFT is modified to develop the Discrete Fourier Transform (DFT), the most practical type of the Fourier transform.
- We look at the properties and limitations of the DFT and its algorithmic cousin, the FFT. We examine the use of Windows to reduce leakage effects due to truncation.
- We examine the application of the DFT/FFT to random signals and the role of auto-correlation function in the development of the power spectrum.
- Lastly, we discuss methods of spectral power estimation. We focus on non-parametric power estimation of stationary random signals using the Periodogram and the Autopower.

If you would like to get updates and notices of error corrections to this book, please email me at charldsp@gmail.com.

And if this book has helped you, please do post a review on Amazon for us. Thank you.
Charan Langton and Victor Levin

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Contents

Chapter 2

Complex Representation of Continuous-Time Periodic Signals

Leonhard Euler
1707 - 1783

Leonhard Euler was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function. Euler is considered to be one of the greatest mathematicians to have ever lived. A student of Johann Bernoulli, Euler was the foremost scientist of his day. Born in Switzerland, he spent his later years at the University of St. Petersburg in Russia. He perfected plane and solid geometry, created the first comprehensive approach to complex numbers. Euler was the first to introduce the concept of $\log x$ *and* e^x *as functions and it was his efforts that made the use of* $e, \imath$ *and* π *the common language of mathematics. Among his other contributions were the consistent use of the trigonometric sine, and cosine functions and the use of a symbol for summation. A father of 13 children, he was a prolific man in all aspects, languages, medicine, botany, geography and physical sciences and has left his mark on our scientific thinking.– From Wikipedia*

Euler's Equation

$$\boxed{e^{j\omega t} = \cos\omega t + j\sin\omega t}$$

This equation is called the Euler's equation. Bertrand Russell and Richard Feynman both gave this equation plentiful praise with words such as "the most beautiful, profound and subtle expression in mathematics" and "the most amazing equation in all of mathematics." This perplexing equation was developed by Euler (pronounced Oiler) in the early 1800s.

The $e^{j\omega t}$ in Euler's equation is a decidedly confusing concept. What exactly is the role of j in $e^{j\omega t}$? We know from algebra that it stands for $\sqrt{-1}$ but what is it doing here with the sine and cosine? Can we even visualize this function?

The complex exponential

The function $e^{j\omega t}$ goes by the name of **complex exponential** (CE). This function is of the greatest importance in signal processing and Fourier analysis. We are going to discuss its conceptual nature and its application to Fourier analysis.

$$e^{j\omega t} = \underline{\cos\omega t + j\sin\omega t} \tag{2.1}$$

Looking at Eq. (2.1), we see the complex exponential on the left side. It is called the **positive complex exponential**, for the simple reason that the exponent of e is positive. On the right of the equal sign, underlined, is the expression containing a sine and a cosine. For now, ignore the complex exponential $e^{j\omega t}$ on the left-hand side and examine only the right-hand side of this equation, containing the sine and cosine waves, with the complex operator j thrown in.

This is a complex function which often means that it is a 3D function. It can be plotted by assigning a particular value to the radial frequency ω, and then for a range of time t, calculating both $\cos\omega t$ and $\sin\omega t$ values. Because the value of ω is held constant, we have three values, the independent time variable, t, and associated sin and cos values, $\cos(\omega t)$ and $\sin(\omega t)$. With these three values, t, $\cos(\omega t)$ and $\sin(\omega t)$, we can create the 3D plot shown in Fig. 2.1. Time is plotted on the x-axis, and y and z axis contain the values of the function $\cos(\omega t)$ and $\sin(\omega t)$. Each plotted helix is for a particular value of frequency, ω. Changing the frequency will change this figure.

The expression for the **negative complex exponential** (negative exponent) is written similar to Eq. (2.1) as:

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t \tag{2.2}$$

We can plot these figures in Matlab using this code.

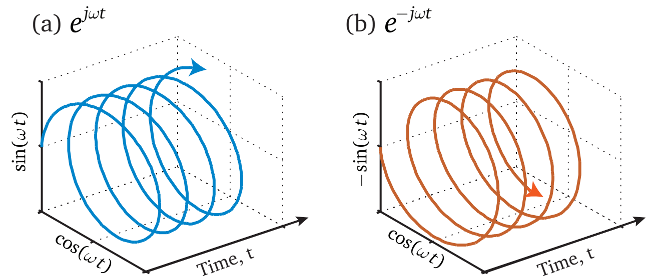

Figure 2.1: $e^{j\omega t}$ when plotted looks like is a helix. It is a 3-D function of three values, time t, $sin(\omega t)$ and $cos(\omega t)$ for a fixed frequency ω. The exponent of the CE indicates direction of advance or movement, (a) positive exponent and (b) negative exponent.

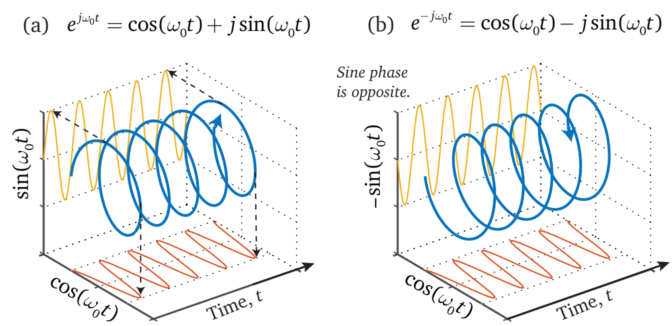

Figure 2.2: The projections of the complex exponential are sinusoids. (a) $e^{j\omega t}$ and its two projections, (b) $e^{-j\omega t}$ and its two projections. Note that the sine wave in (b) has different phase than one for the positive CE in (a). That is the only difference between the two CEs.

```
t = 0:0.01:5;
y=exp((j*(2*pi))*t); % positive exponent
subplot(1,2,1);
plot3(t,real(y),imag(y), 'linewidth', 1);
subplot(1,2,2);
y2=exp((-j*(2*pi))*t); % negative exponent
plot3(t,real(y2),imag(y2), 'linewidth', 1);
```

Projections of a complex exponential

As the CE is a complex function, we examine its projections on the real and the imaginary axes. In Fig. 2.2(a) we plot the projections of the helix on the *Real* and the *Imaginary* planes. In Cartesian terms, these would be called the (X,Y) and (Z,Y) planes. The projections of the complex exponential on these two planes are sine and cosine waves. The Real projection of the complex exponential is a cosine wave and Imaginary projection is a sine wave.

These projections can be plotted using the following Matlab code.

```
t = 0:0.01:5;
y=exp((j*(2*pi))*t); % Positive exponential
subplot(1,2,1);
plot3(t,real(y),imag(y), 'linewidth', 1);
th2 = linspace(-2,-2, length(real(y))); hold on;
plot3(t, real(y), th2,'linewidth', .5); hold on;
th = linspace(2,2, length(real(y)))
plot3(t, th, imag(y), 'linewidth', .5)
subplot(1,2,2);
y2=exp((-j*(2*pi))*t); % Negative exponential
plot3(t,real(y2),imag(y2), 'linewidth', 1);
th2 = linspace(-2,-2, length(real(y2))); hold on;
plot3(t, real(y2), th2,'linewidth', .5); hold on;
th = linspace(2,2, length(real(y2)))
plot3(t,th, imag(y2), 'linewidth', .5)
```

For the negative exponent, or the so-called negative complex exponential, the projection, sine wave, is flipped 180° degrees from the positive exponential, as shown in Fig. 2.2(a). Often this exponential is referred to as having a negative frequency; however, it is not really the frequency that is negative. From the definition of the negative exponent exponential in Eq. (2.2), we see that negative sign of the exponential results in the imaginary projection, the sine wave doing a 180° phase change, or equivalently being multiplied by -1.

The real part of the positive CE as well as the negative CE is a positive cosine wave.

$$\text{Re}\left(e^{j\omega t}\right) = \cos\omega t \qquad \text{Re}\left(e^{-j\omega t}\right) = \cos\omega t$$

The imaginary part of the positive CE is a positive sine but is negative for the negative CE.

$$\text{Im}\left(e^{j\omega t}\right) = \sin\omega t \qquad \text{Im}\left(e^{-j\omega t}\right) = -\sin\omega t$$

While the negative exponential has as its imaginary part a negative sine wave, the positive exponential has a positive sine as its imaginary part. The real part, which is a cosine, is same for both. We do not see any negative frequencies here, an idea generally associated with the negative complex exponential.

Adding and subtracting the complex exponentials, $e^{j\omega t}$ and $e^{-j\omega t}$, and then after a little rearrangement, we get these new ways of expressing a sine and a cosine.

$$\begin{aligned} \frac{1}{2}\left(e^{+j\omega t} + e^{-j\omega t}\right) &= \frac{1}{2j}\left(\cos\omega t + \cancel{j\sin\omega t} + \cos\omega t - \cancel{j\sin\omega t}\right) \\ &= \cos(\omega t) \end{aligned} \tag{2.3}$$

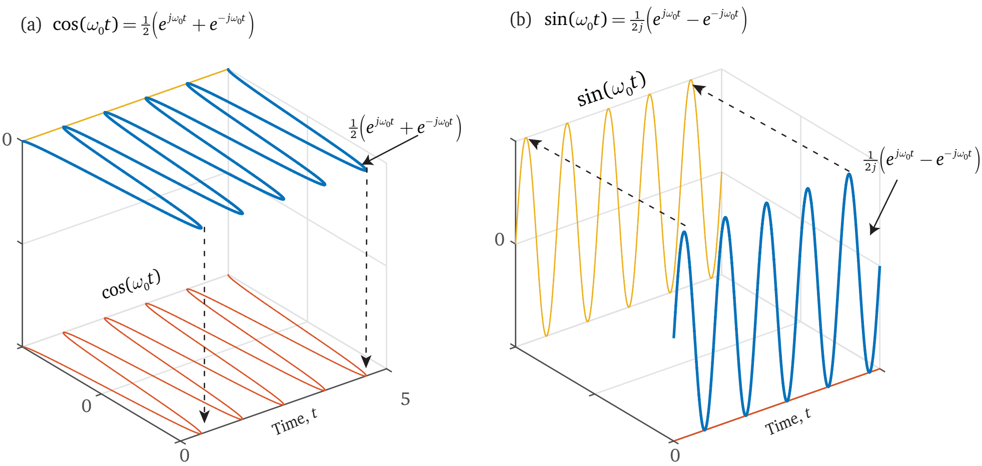

Figure 2.3: (a) Plotting $(e^{+j\omega t} + e^{-j\omega t})/2$ gives a cosine wave with zero projection on the imaginary plane (b) Plotting $(e^{+j\omega t} - e^{-j\omega t})/2j$ gives us a sine wave with zero projection on the real axis.

$$\begin{aligned} \frac{1}{2j}\left(e^{+j\omega t} - e^{-j\omega t}\right) &= \frac{1}{2j}\left(\cancel{\cos\omega t} + j\sin\omega t - \cancel{\cos\omega t} + j\sin\omega t\right) \\ &= \sin(\omega t) \end{aligned} \tag{2.4}$$

Let's see graphically what Eq. (2.3) and Eq. (2.4) look like. When we plot the two composite exponentials, we get the two plots as shown in Fig. 2.3. Fig. 2.3(a) shows that this composite exponential has a real projection of a cosine and Fig. 2.3(b), only the sine. The helix is gone, it has collapsed into a cosine or a sine. Hence, the sine and cosine can be said to be composed of complex exponentials. So a CE is composed of a cosine and a sine and conversely a sinusoid is also composed of two CE.

We are so used to thinking of sine and cosine as sort of atomic functions, divinely given that it seems hard to believe that they can be created from other functions. But Eq. (2.3) and Eq. (2.4) tell us that both sine and cosine can be created by adding complex exponentials. This is a case of two 3D functions coming together to create a 2D sinusoid.This sounds strange but it's actually not an unfamiliar concept. We can add two 2D functions and get a 1D function. An example is when we add a sine and a 180° shifted sine, we get a straight line, a 1D function. So a 2D function created by two 3D functions should not be a big stumbling block.

The Sinusoids

So how did Euler's equation come about and why is it so important to signal processing? We will try to answer that by first looking at Taylor series representations of the exponential e^x, sines, and the cosines. The Taylor series expansion for the two sinusoids is

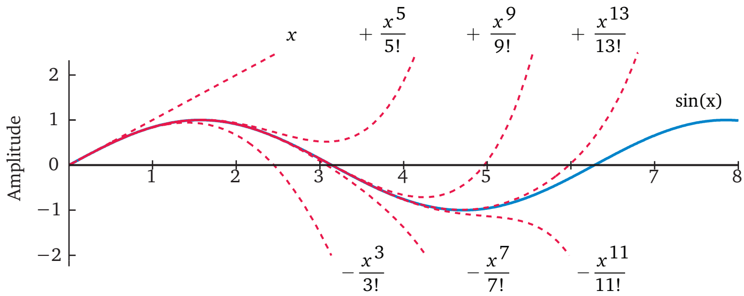

Figure 2.4: Sine wave as a sum of many exponentials of different weights.

given in 2D by the infinite series as:

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \end{aligned} \tag{2.5}$$

Note that each one of these series is composed of many individual exponential functions. Thus, sine waves really are composed of exponentials! However, these are ***real*** exponentials that are nonperiodic and not the same thing as the complex exponentials. Complex exponentials are a special class of real exponentials and are used as an alternate to the sinusoids in Fourier analysis, as they are periodic and offer ease of expression and calculation, which is not obvious when we encounter them at first.

Real exponentials are used in Laplace analysis as the basis set, instead of the complex exponentials we use in Fourier analysis. Real exponentials are more general than complex exponentials and allow analysis of nonperiodic and transient signals. Laplace analysis is a general case of which Fourier is a special case applicable only to periodic or *mostly* periodic signals.

The Taylor series expansion for a *real* exponential e^x gives this expression

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \tag{2.6}$$

Both of these equations, Eq. (2.5) and Eq. (2.6) are straightforward concepts after you have digested them a bit. And indeed, if these functions were plotted, we would get just what we are expecting, the exponential of e and the sinusoids. How close our plots come to the continuous function depends on the number of terms that are included in the summation.

The similarity between the exponential and the sinusoids series in Eq. (2.5) and Eq. (2.6) shows clearly that there is a relationship here. Now let us change the exponent in Eq. (2.6) from x to $j\theta$. Note that the term θ is used here instead of ωt simply to keep the equation concise. Now we have by simple substitution, the $e^{j\theta}$ series:

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \dots \tag{2.7}$$

We know that $j^2 = -1$ and $j^4 = 1$, $j^6 = -1$, etc., substituting these values, we rewrite this series as

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{j\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{j\theta^7}{7!} + \dots \tag{2.8}$$

We can separate out every other term with j as a coefficient to create a two-part series, one without the j and the other with

$$\begin{aligned} e^{j\theta} &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \qquad \text{This is cosine} \\ &+ j\theta - \frac{j\theta^3}{3!} + \frac{j\theta^5}{5!} - \frac{j\theta^7}{7!} + \dots \qquad \text{This is } j \text{ times sine} \end{aligned} \tag{2.9}$$

It can be seen that the first part of the series is a cosine per Eq. (2.5) and the second part with j as its coefficient is the series for a sine wave. Hence, we see that Euler's equation is really quite understandable! It came from the Taylor series. The concept of a Taylor series was formulated by the Scottish mathematician James Gregory but introduced into the world by the English mathematician Brook Taylor in 1715, so he gets his name on it instead.

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

We can now derive some interesting results from the Euler's equation, such as the following. By setting $\theta = \pi/2$, we can show that

$$\begin{aligned} e^{j\pi/2} &= \cancel{\cos(\pi/2)} + j\sin(\pi/2) \\ &= 0 + j \cdot 1 \\ &= j \end{aligned}$$

By setting $\theta = 3\pi/2$, we can show that

$$\begin{aligned} e^{j3\pi/2} &= \cancel{\cos(3\pi/2)} + j\sin(3\pi/2) \\ &= 0 + j \cdot -1 \\ &= -j \end{aligned}$$

And an another interesting result:

$$e^{j\pi} = \cos(\pi) + j\sin(\pi) = -1$$

From this, we can write this amazing looking equation!

$$e^{j\pi} + 1 = 0$$

So this complex looking function is not so complicated after all. These are some of the useful properties of a CE that we will be using. The function maintains a wondrous and mysterious quality, with added tinge of fear for some. However, we need to get over our fear of this equation and learn to love it. But, the question now is why bring up the Euler's equation in context of Fourier analysis at all? Why all this rigmarole about the complex exponential, why are sines and cosines not good enough? They are certainly easier to visualize.

In Fourier analysis, we computed the coefficients of sines and cosines (the harmonics) separately. In Chapter 1, we discussed the three different formulations of the Fourier series using sines and cosines, only with cosines, and with complex exponentials. Fourier analysis using the trigonometric form is not easy in practice. Trig functions are easy to understand but hard to manipulate. In fact, adding and multiplying them is a pain. On the other hand, doing math with exponentials is considerably easier. (See examples in Appendix A.)

Using a single exponential representing both sinusoids can simplify calculations in Fourier series. This is the main advantage of switching to complex exponentials in using the complex form of the Fourier series. The math looks hard but is actually easier. However, complex exponentials bring with them some conceptual difficulties, which is that they are hard to visualize and are confusing at first.

Typically, when we decompose something, we do it into a simpler form but here seemingly a more complex form is being employed. A simpler quantity, a cosine wave is now decomposed into two complex functions. However, the net result is that it will make Fourier analysis simpler. We will go from simplicity to complexity and then to simplicity again.

Let us take this sinusoid that has a phase term to complicate things and present it in complex form.

$$\begin{aligned} x(t) &= A\cos(\omega t + \theta) \\ &= \frac{A}{2}e^{j(\omega t+\theta)} + \frac{A}{2}e^{-j(\omega t+\theta)} \\ &= \frac{A}{2}e^{j\omega t}e^{j\theta} + \frac{A}{2}e^{-j\omega t}e^{-j\theta} \end{aligned}$$

In the last row, we separate the exponential into its powers. If we expand this expression into trigonometric domain using Euler's equation, we see that indeed we do get back the trigonometric cosine wave we started with.

$$= \frac{A}{2}\Big(\cos(\omega t+\theta) + \cancel{j\sin(\omega t+\theta)} + \cos(\omega t+\theta) - \cancel{j\sin(\omega t+\theta)}\Big)$$
$$= A\cos(\omega t+\theta)$$

Fourier Series Representation using Complex Exponentials

In Chapter 1, we used trigonometric harmonics (the sine and cosine) as a basis set to develop the Fourier series representation. The target signal was "mapped" onto a k set of sinusoidal harmonics, such as these based on fundamental frequency of ω_0.

$$S = \left[\,\sin\omega_0 t,\ \cos\omega_0 t,\ \sin 2\omega_0 t,\ \cos 2\omega_0 t, \ldots, \cos k\omega_0 t, \ldots,\,\right]$$

A set of complex exponentials given by set S, can then be used alternately as a basis set for creating a complex (but preferred) form of the Fourier series.

$$S = \left[\,\ldots, e^{-3j\omega_0 t},\ e^{-2j\omega_0 t},\ e^{-1j\omega_0 t},\ 1\,, e^{1j\omega_0 t}, \ldots, e^{kj\omega_0 t}, \ldots\right]$$

The big difference between these two forms is that whereas for the trigonometric form, the index k is strictly positive, for the complex exponential form, it covers both positive and negative integers. This is an important difference as we shall see in this chapter. These complex exponentials are an orthogonal set, making them easy to separate from each other. This is the main reason why we pick orthogonal signals to represent something. Just as our 3D world is defined along three orthogonal axes, X, Y, and Z, our signals can be similarly projected on a K dimensional orthogonal set. We can not imagine these K orthogonal axes, but we know mathematically that they exist.

As mentioned earlier, the Fourier series is a sum of weighted sinusoids. By weighted we mean that each sinusoid has its own amplitude and starting phase. The time is continuous but frequency resolution is not, frequency takes on discrete harmonic values. If the fundamental frequency is ω_0, then each ω_k is an integer multiple of ω_0 and is discrete no matter how large k gets. The "distance" between each harmonic remains the same, ω_0.

Let us examine the trigonometric Fourier series equation again.

$$f(t) = a_0 + \sum_{k=1}^{K} a_k \cos(\omega_k t) + \sum_{k=1}^{K} b_k \sin(\omega_k t) \tag{2.10}$$

The coefficients a_0, a_k, b_k (which we call the trigonometric coefficients) are given by (from Chapter 1):

$$\begin{aligned} a_0 &= \frac{1}{T_0}\int_0^{T_0} f(t)\,dt \\ a_k &= \frac{2}{T_0}\int_0^{T_0} f(t)\cos(k\omega_0 t)\,dt \\ b_k &= \frac{2}{T_0}\int_0^{T_0} f(t)\sin(k\omega_0 t)\,dt \end{aligned} \tag{2.11}$$

The presence of an integral tells us that time is continuous. Now substitute Eq. (2.3) and Eq. (2.4), as the definition of sine and cosine into Eq. (2.10), to get

$$f(t) = a_0 + \sum_{k=1}^{K} \frac{a_k}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) + \sum_{k=1}^{K} \frac{b_k}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right) \tag{2.12}$$

Rearranging Eq. (2.12) so that each complex exponential is separated, we get

$$f(t) = a_0 + \sum_{k=1}^{\infty} \frac{1}{2}(a_k - jb_k)e^{jk\omega_0 t} + \sum_{k=1}^{\infty} \frac{1}{2}(a_k + jb_k)e^{-jk\omega_0 t} \tag{2.13}$$

Let us redefine two new terms, called the complex coefficients, A_k and B_k as:

$$A_k = \frac{1}{2}(a_k - jb_k) \tag{2.14}$$

$$B_k = \frac{1}{2}(a_k + jb_k) \tag{2.15}$$

Substituting these new coefficients into Eq. (2.13), we get this representation.

$$f(t) = a_0 + \sum_{k=1}^{\infty} A_k e^{jk\omega_0 t} + \sum_{k=1}^{\infty} B_k e^{-jk\omega_0 t} \tag{2.16}$$

By substituting the complex exponential for the sinusoids, we can rewrite the trigonometric coefficients from Eq. (2.11) as:

$$a_k = \frac{2}{T_0}\int_0^{T_0} f(t)\frac{1}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) dt \tag{2.17}$$

$$b_k = \frac{2}{T_0}\int_0^{T_0} f(t)\frac{1}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right) dt \tag{2.18}$$

These can be expanded as follows.

$$a_k = \frac{1}{T_0}\int_0^{T_0} f(t)e^{jk\omega_0 t}dt + \frac{1}{T_0}\int_0^{T_0} f(t)e^{-jk\omega_0 t}\,dt \tag{2.19}$$

$$b_k = \frac{1}{T_0}\int_0^{T_0} f(t)e^{jk\omega_0 t}dt - \frac{1}{T_0}\int_0^{T_0} f(t)e^{-jk\omega_0 t}\,dt \tag{2.20}$$

You can see that the trigonometric coefficient is split into two parts now, one for each of the exponentials. With some manipulation and using some simple complex math, we can show that the new complex coefficients are given by:

$$A_k = \frac{1}{T_0}\int_0^{T_0} f(t)e^{-jk\omega_0 t}\,dt \tag{2.21}$$

$$B_k = \frac{1}{T_0}\int_0^{T_0} f(t)e^{jk\omega_0 t}\,dt \tag{2.22}$$

Hence, A_k can be thought of as the coefficient of the positive exponential and B_k the coefficient of the negative exponential, analogous to the trigonometric coefficients belonging to the cosine and the sine harmonics, a_k and b_k. The equivalence between the complex coefficients and the trigonometric coefficients takes the exact same form as the Euler equation. We will see in the example problems the effect of the trigonometric coefficients splitting into two parts. The total magnitude at a particular frequency is the same for both complex and the trigonometric forms, but appears split in the spectrum because of the sign of the index. We still have to contend with the term a_0, the DC component in Eq. (2.16). We can get rid of this DC term by including it inside the summation. We do that by expanding the range of the index k from 0 to ∞ instead of starting at 1. Rewrite Eq. (2.16) as

$$\boxed{f(t) = \sum_{k=0}^{\infty} A_k e^{jk\omega_0 t} + \sum_{k=0}^{\infty} B_k e^{-jk\omega_0 t}} \tag{2.23}$$

Going forward, Eq. (2.23) can be simplified even further by extending the range of coefficients from $-\infty$ to ∞. Since the index now expands from positive domain to negative, we no longer need the negative CE in Eq. (2.23). Now both terms can be combined into one positive CE with a two-sided index so that we get a much more compact and elegant equation for the Fourier series. There is no negative exponential in this equation because the index takes care of that. And here is a much shorter equation for the Fourier series in the complex domain.

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}} \tag{2.24}$$

The coefficient C_k in Eq. (2.24) is given by

$$\boxed{C_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)e^{-jk\omega_0 t}\, dt} \tag{2.25}$$

The equations Eq. (2.25) and Eq. (2.24) are called the complex form of the Fourier series. They are rigorously related to the trigonometric form. The magnitude calculated using the trigonometric form is exactly the same as the magnitude from this form. These two equations are the most used form of the Fourier series. The complex coefficients, C_k are of course equivalent to the trigonometric coefficients by this relationship.

$$\begin{aligned} C_k &= \frac{1}{2}(a_k - jb_k) \text{ if } k \geq 0 \\ &= \frac{1}{2}(a_k + jb_k) \text{ if } k < 0 \end{aligned} \tag{2.26}$$

As explained in Chapter 1, for the trigonometric form, the index k is always positive and therefore the spectrum for the Fourier series using the trigonometric form is one-sided. The x-axis for the one-sided spectrum is plotted against frequency starting at zero frequency or the "positive" fundamental frequency, which means that all k integer multiples of the fundamental frequency are positive. Because the index is positive, all frequencies are said to be positive. Now the index is allowed to be negative and this gives rise to the idea that the frequency has become negative, an effect of doing math in the complex domain. If the two forms are equivalent, does the frequency actually become negative when we use complex exponentials? With complex formulation, the index k spans from $-\infty$ to $+\infty$. We start with the negative index, go through calculations of all negative exponent exponentials and then the positive ones. Note that at no point is the fundamental frequency ever negative. Hence it is not the frequency of the exponential that is negative but *just* the index. The exponential with the negative index k is different from the positive exponential in that the sign of the imaginary part is negative. There is nothing here that implies that the frequency itself is negative.

$$\begin{aligned} e^{+jk\omega_0 t} &= \cos(k\omega_0 t) + j\sin(k\omega_0 t) \\ e^{-jk\omega_0 t} &= \cos(k\omega_0 t) - j\sin(k\omega_0 t) \end{aligned} \tag{2.27}$$

We now equate this form with the trigonometric form that seemingly had only positive frequencies. To accomplish this, we look at what it takes to represent a sine and a cosine using complex exponentials.

$$\begin{aligned}\cos(k\omega_0 t) &= \frac{1}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right)\\ \sin(k\omega_0 t) &= \frac{1}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right)\end{aligned} \tag{2.28}$$

We require both a negative-index exponential and a positive-index exponential to describe both the sines and cosines. Where index k is always positive on the left-hand side of this equation, it is both negative and positive on the right side. This traps us into thinking that frequency has changed sign. While in trigonometric form a positive index is enough to fully and completely represent the signal, in complex form it takes a double-sided index. The spectrum is plotting the product of the index and the frequency, $(k\omega)$, and not just the frequency, ω, on the x-axis. But we very quickly lose sight of this fact. We start talking about positive and negative frequencies because we confuse the *range* of the index with the sense of the frequency.

Double-sided spectrum

This gives rise to the **double-sided spectrum** which spans from values of k from -∞ to $+\infty$. In this type of spectrum, the x-axis represents frequency, but that isn't necessarily so. What we actually plot is the product of the harmonic index and the frequency. Calling it frequency gives us some intuitive comfort but then we have to worry about what a negative frequency means. When we say *negative frequency*, we have, in fact, unknowingly converted a complex idea into simple everyday language. Because of the plotting convention, the sign of the index is often forgotten and the axis is referred to as the frequency axis, spanning both positive and negative domains. In this book, we maintain that there is no such thing as a negative frequency. The idea comes from confusion caused by what the x-axis represents in the complex domain.

The complex coefficient values are one-half of what they are calculated in the trigonometric domain. Sometimes students think that this is because the frequency is being split into two parts, a negative part and positive part with each getting half the coefficient. This is how some authors try to explain the conundrum of positive and negative frequency in relation to Fourier analysis. However, they are just trying to explain a plotting convention. The real story is that we are not plotting frequency on the x-axis of a spectrum but the term $\pm k\omega_0$. First we call this "frequency" and then we try to explain this error. A CE is a helical function with a sense of rotational direction in addition to its frequency. The index k indicates this direction.

The reason the amplitude values are split in half can be explained intuitively. We have let the index k go from $-\infty$ to $+\infty$, thus, each frequency is now multiplied by both a positive k and a negative k. However, in reality, each frequency has only a certain amount of finite energy, so to make it all work out, the energy of this frequency is split between these two indices. Each index gets half.

There are certain things that are defined only as positive quantities, such as volume, mass, age, etc., and frequency as a physical property is one of those things. It is never negative. In complex domain, we look at the signal in three dimensions, and hence a signal has a frequency and a rotational direction. So the term frequency, although a scalar, may be thought to include this other rotational parameter.

Example 2.1. Compute the complex coefficients of a cosine wave.

$$\begin{aligned} f(t) &= A\cos(\omega_0 t) \\ &= \frac{A}{2}e^{j(k=1)\omega_0 t} + \frac{A}{2}e^{j(k=-1)\omega_0 t} \end{aligned} \tag{2.29}$$

This example is so simple that we can easily deduce the trigonometric coefficients just by looking at the expression. In fact, the equation is itself a perfect Fourier representation. The first thing we do is to write the equation in terms of its Euler form using Eq. (2.28). This form has the two CEs with the appropriate coefficients. The complex coefficients of the sinusoids are of magnitude $A/2$ located at $k=1$ and $k=-1$. We plot the coefficients, C_k, in Fig. 2.5 as the single-sided spectrum as well as the exponential coefficients, C_k, as the double-sided form. The x-axis variable is $k\omega$. Since, ω_0 is a constant, we are really plotting, k, the index. Note, it is not the frequency that is negative but the harmonic index k. However, in a typical plot, the x-axis is labeled as frequency. In these plots, we have labeled it specifically as what it is, the term $k\omega_0$.

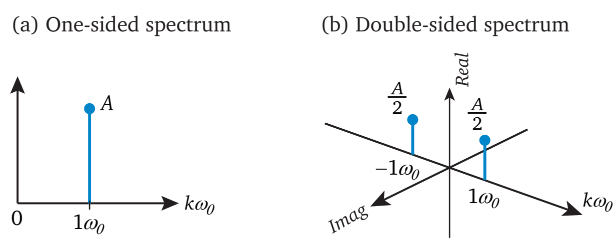

Figure 2.5: The spectrum of $A\cos\omega t$. The trigonometric form gives us a one-sided spectrum with one component located at ω. The complex form shows two components of half the total amplitude in the real plane. The vertical axis is amplitude for both forms. The total energy in both forms is the same, despite being split in exponential form in two parts.

Example 2.2. Compute complex coefficients of a sine wave

$$\begin{aligned} f(t) &= A\sin\omega_0 t \\ &= \frac{A}{2j}e^{j(k=1)\omega_0 t} - \frac{A}{2j}e^{j(k=-1)\omega_0 t} \end{aligned} \tag{2.30}$$

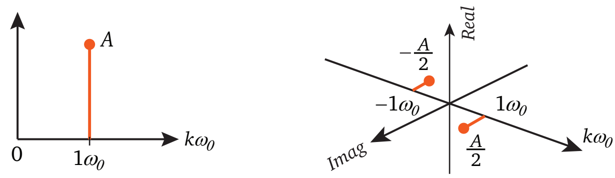

Figure 2.6: The spectrum of $A \sin \omega t$. The trigonometric form gives us a one-sided spectrum with one component located the ω. The complex form shows two components of half the total amplitude of opposite signs in the imaginary plane.

This example is just the same as the cosine example. The single-sided spectrum is easy. It is simply a harmonic of magnitude A and located at $k = 1$ just as it is for the cosine wave in Ex. 2.1. In Eq. (2.30), we write the complex form based on Eq. (2.28), with the amplitudes of the two complex exponentials as $A/2$ and $-A/2$ located at $k = \pm 1$. While the amplitudes were positive for cosine in Ex. 2.1, here they are of opposite signs.

However, there is also the dreaded j in the denominator of Eq. (2.30). What to do with this? The presence of j tells us that the coefficients are on the imaginary axis, so they are to be plotted on the imaginary plane, right-angle to the plane on which a cosine lies. Drawn in 2D form it has no computational effect, only that the horizontal axis is called the imaginary axis. However, when there are both cosines and sine waves present in a signal, the coefficients of these two have to be combined not linearly but as a vector sum as seen in Fig. 2.7. This is because the harmonics are orthogonal to each other. When plotting the magnitude, it no longer falls in just the *Real* or the *Imaginary* planes but somewhere in between. In the next example, we combine both of these sinusoids in a single function.

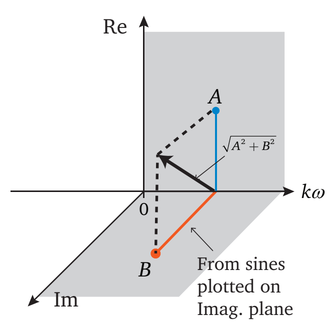

Figure 2.7: Magnitude of the resultant vector if a signal contains both a sine and a cosine.

Example 2.3. Compute the coefficients of $f(t) = A(\cos \omega t + \sin \omega t)$.

We can write this wave as

$$f(t) = \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t} + \frac{A}{2j}e^{j\omega t} - \frac{A}{2j}e^{-j\omega t} \qquad (2.31)$$

Once again, we convert this function to its complex form using Eq. (2.28). From this form, we can pick out the trigonometric coefficients easily. It is simply A for the cosine and A for sine with magnitude equal to square root of $\sqrt{2}A$ located at $\omega = 1$.

The coefficients of the complex exponentials, too can be obtained by looking at the coefficients of the two exponentials in Eq. (2.31). The $e^{j\omega t}$ exponential has two coefficients, at 90° to each other, each of magnitude $A/2$. The vector sum of these is $A/\sqrt{2}$. Same for the negative exponential, except the amplitude contribution from the sine is negative. However, the vector sum or the magnitude for both is the same and always positive. This is shown in Fig. 2.8(c) drawn in a more conventional style showing only the vector sum. Note that the total energy (sum of magnitudes) of the one-sided spectrum is exactly the same as that of complex.

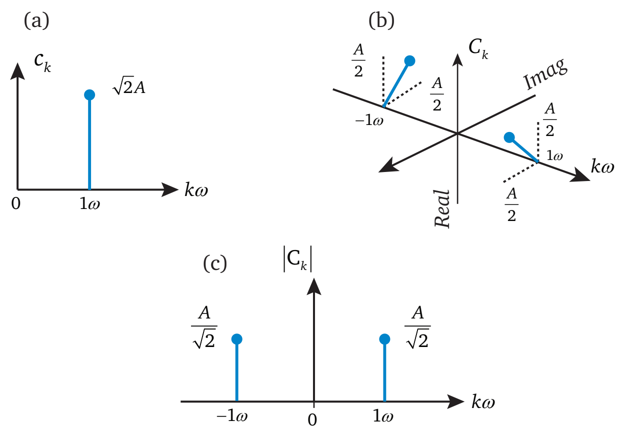

Figure 2.8: Amplitude spectrum of $A\sin\omega t + A\cos\omega t$.

Example 2.4. Compute coefficients of the complex signal $f(t) = A(\cos\omega t + j\sin\omega t)$.

This function is different from the one in Ex. 2.3 in that the sin is located in the imaginary plane. This is a complex signal. We can write this in CE form as:

$$f(t) = \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t} + \frac{A}{2}e^{j\omega t} - \frac{A}{2}e^{-j\omega t} = Ae^{j\omega t}$$

The coefficients from the sine and the cosine for the negative exponential cancel each other. On the positive side, the contribution from sine and cosine are coincident and add. Therefore, we see a single value at the positive index of $k = 1$ only. For this signal, both the single-sided and double-sided spectrum are identical. This is a surprising and perhaps a counter-intuitive result.

Important observation: *Only real signals have symmetrical spectrum about the origin. Complex signals do not.*

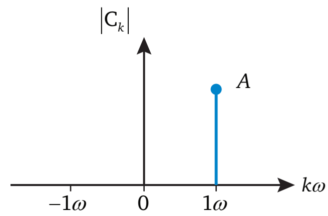

Figure 2.9: Double-sided spectrum of $A\cos\omega t + jA\sin\omega t$.

Example 2.5. Compute the coefficients of this signal, $f(t) = A$.

We can write this function as an exponential of zero frequency.

$$\begin{aligned} f(t) &= A\cos(\omega = 0)t \\ &= \frac{A}{2}e^{j(\omega=0)t} + \frac{A}{2}e^{-j(\omega=0)t} \\ &= A. \end{aligned}$$

The trigonometric coefficient is A at $\omega = 0$. For the complex representation, we get two complex coefficients, both of amplitude $A/2$ but both at $k = 0$ so their sum is A, that is exactly the same as in the trigonometric representation. The function $f(t)$, a constant, is a non-changing function of time and we classify it as a DC signal. The DC component, if any, always shows up at the origin. The single-sided and double-sided spectrum here are same as well. This signal has only one component at $\omega = 0$; hence, it has only one, coefficient a_0, all the others are zero. Therefore, that is what we are seeing in Fig. 2.10. Just the a_0 coefficient plotted.

Important observation: *A component at zero frequency means that the signal is not zero-mean*.

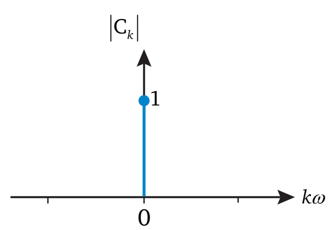

Figure 2.10: Double-sided spectrum of a constant signal of amplitude A. A constant signal is same as a DC value, hence, its spectrum always appears as an impulse at the origin.

Example 2.6. Compute the coefficients of $x(t) = 2\cos^2(\omega t)$.

We express this function in complex form as:

$$x(t) = 2\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right)^2$$
$$= 1 + \frac{1}{2}e^{j2\omega t} + \frac{1}{2}e^{-j2\omega t}$$

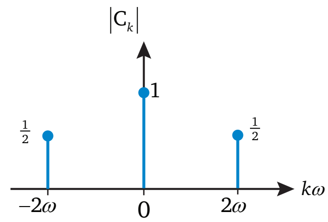

Figure 2.11: Double-sided amplitude spectrum of $2\cos^2(\omega t)$.

The coefficients become obvious when we put the trigonometric form into the complex exponential form. There is just one frequency, 2ω and it has an amplitude of $\frac{1}{2}$ at k of ± 2. The coefficient of zero frequency is 1.

Observation: *A squared signal by definition is always positive so the spectrum has a zero-frequency component at the origin.*

Example 2.7. Compute the coefficients of $x(t) = 2\cos(\omega t)\cos(2\omega t)$.

We can express this signal in complex form by making use of this trigonometric identity: $2(\cos(a)\cos(b)) = \cos(a - b) + \cos(a + b)$.

$$x(t) = \cos(\omega t) + \cos(3\omega t)$$
$$= \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t} + \frac{1}{2}e^{j3\omega t} + \frac{1}{2}e^{-j3\omega t}$$

Of course, doing this in trigonometric form would have been just as easy; however that is not always true. We draw the spectrum as in Fig. 2.12.

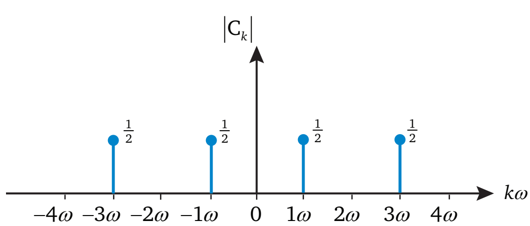

Figure 2.12: Double-sided amplitude spectrum of $2\cos(\omega t)\cos(2\omega t)$.

This signal has four coefficients, $a_{\pm 1}$ and $a_{\pm 3}$; all the others are zero. So, that is what we are seeing in Fig. 2.12.

Example 2.8. Compute the complex coefficients of this real signal.

$$f(t) = \sin(4\pi t) + 0.8\cos(8\pi t) + 0.3\sin(14\pi t)$$

We want all these frequencies to fall on a harmonic, so we select as the fundamental frequency, $f = 1$. The signal has six components, at $k = \pm 2, \pm 4, \pm 7$. We can write this equation in complex form as:

$$\begin{aligned} f(t) = & \frac{1}{2} e^{j2\pi(k=2)t} - \frac{1}{2} e^{j2\pi(k=-2)t} \\ & + \frac{0.8}{2j} e^{j2\pi(k=4)t} + \frac{0.8}{2j} e^{j2\pi(k=-4)t} \\ & + \frac{0.3}{2j} e^{j2\pi(k=7)t} - \frac{0.3}{2} e^{j2\pi(k=-7)t} \end{aligned}$$

The contributions at $k = 2$ comes only from a sine, at $k = 4$ from a cosine and at $k = 7$ only from a sine. Note, we plot these on the same line at full magnitude as if j does not exist in the equation. (We will drop mentioning the index k and call it frequency to fall in line with common usage. However, note that it is this sloppiness in terms that causes us to question our sanity and start asking: what is a negative frequency?)

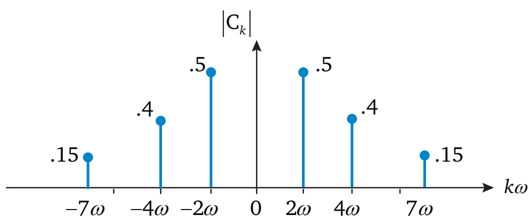

Figure 2.13: Two-sided symmetrical magnitude spectrum.

This signal has six coefficients, $a_{\pm 2}$, $a_{\pm 4}$ and $a_{\pm 7}$. All the others are zero. So that is what we are seeing here in Fig. 2.13. Just those three coefficients plotted on each side, with half the amplitudes from the time domain equation.

Example 2.9. Compute the complex coefficients of this real signal with phase terms. Then compute its power spectrum.

$$x(t) = 3 + 6\cos(4\pi t + 2) + 4j\sin(4\pi t + 3) - 3j\sin(10\pi t + 1.5)$$

We convert this to the complex form as

$$\begin{aligned} x(t) &= 3 + \left(3e^{j4\pi t}e^{j2} + 3e^{-j4\pi t}e^{-j2}\right) + \left(2e^{j4\pi t}e^{j3} - 2e^{-j4\pi t}e^{-j3}\right) \\ &\quad - \left(3e^{j10\pi t}e^{j1.5} - 3e^{-j10\pi t}e^{-j1.5}\right) \\ &= 3 + e^{j4\pi t}\left(3e^{j2} + 2e^{j3}\right) + e^{-j4\pi t}\left(3e^{-j2} - 2e^{-j3}\right) + e^{j10\pi t}\left(1.5e^{j1.5}\right) \\ &\quad + e^{-j10\pi t}\left(1.5e^{-j1.5}\right) \end{aligned}$$

The magnitudes of the exponentials come from the terms in parenthesis. To add, we need to convert them first to rectangular form as follows. (see Appendix A). The CE $e^{j4\pi t}$ has the following coefficients:

$$e^{j4\pi t} \to \left(3e^{j2} + 2e^{j3}\right)$$
$$\Rightarrow \left|3e^{j2} + 2e^{j3}\right| = \sqrt{\left(3\cos(2) + 2\cos(3)\right)^2 + \left(3\sin(2) + 2\sin(3)\right)^2}$$
$$= 4.414$$

Similarly, the coefficient of the negative exponential is

$$e^{-j4\pi t} \to \left(3e^{-j2} - 2e^{-j3}\right)$$
$$\Rightarrow \left|3e^{-j2} - 2e^{-j3}\right| = \sqrt{\left(3\cos(2) - 2\cos(3)\right)^2 + \left(3\sin(2) + 2\sin(3)\right)^2}$$
$$= 2.552$$

We draw the spectrum in Fig. 2.14 and note that the spectrum is not symmetric because the signal is complex.

Important Observation: *Most signals we work with are complex and their spectrum are rarely symmetrical.*

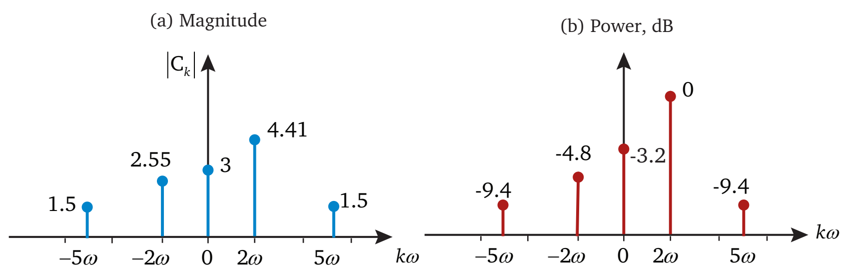

Figure 2.14: (a) Two-sided Magnitude spectrum of a complex signal, (b) its power spectrum computed by squaring each component and converting to dBs.

Power spectrum

You are familiar with this expression of power from circuits. The power is defined as:

$$P = V^2/R$$

Here, V is the voltage or the amplitude of the signal and R the resistance. Assume that R is equal to 1.0; in this case, the normalized power is equal to the voltage squared. If we square the peak voltage, we get peak power, and if we square mean voltage, then mean power is obtained. This idea is exactly the same as Parseval's theorem, which states that the power in a particular harmonic is equal to the square of its amplitude. So, for this particular example, to obtain the power spectrum, we just square each magnitude, convert it into dBs, and then normalize for maximum power. The result is shown in Fig. 2.14(b).

Now a difficult but an important example, a periodic signal of square pulses.

Example 2.10. Compute the Fourier coefficients of a periodic signal consisting of square pulses. The square pulse is of amplitude $1v$ that lasts τ seconds and repeats every T seconds. (Note that τ and T are different and independent quantities.)

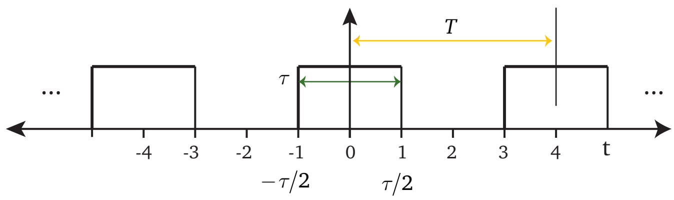

Figure 2.15: A square wave of period T and duty cycle $2/T$.

First, we compute the coefficients for a general case for pulse time equal to τ seconds and repeat time, or the period of the wave, equal to T seconds. The term τ/T is called the duty cycle of the wave.

$$C_k = \frac{1}{T}\int_{-\tau/2}^{\tau/2} 1 \cdot e^{-j2k\pi f_0 t}\, dt$$

Note that outside of $-\tau/2 < t < \tau/2$, the function is zero. This integral is given by

$$\begin{aligned} C_k &= \frac{1}{T}\frac{e^{-(j2\pi k/T)t}}{-j2\pi k/T}\Bigg|_{-\tau/2}^{\tau/2} \\ &= \frac{1}{T}\left(\frac{e^{(j2\pi k/T)\frac{\tau}{2}} - e^{(-j2\pi k/T)\frac{\tau}{2}}}{j2}\right)\frac{1}{\pi k/T} \\ &= \frac{\tau}{T}\frac{\sin(k\pi\tau/T)}{k\pi(\tau/T)} \end{aligned}$$

Replacing the duty cycle term τ with term r, the equation becomes easier to understand.

$$C_k = r\frac{\sin(k\pi r)}{k\pi r} = r\text{sinc}(k\pi r).$$

Let us set the duty cycle of this signal to 0.5.

$$r = \frac{\tau}{T} = \frac{1}{2}$$

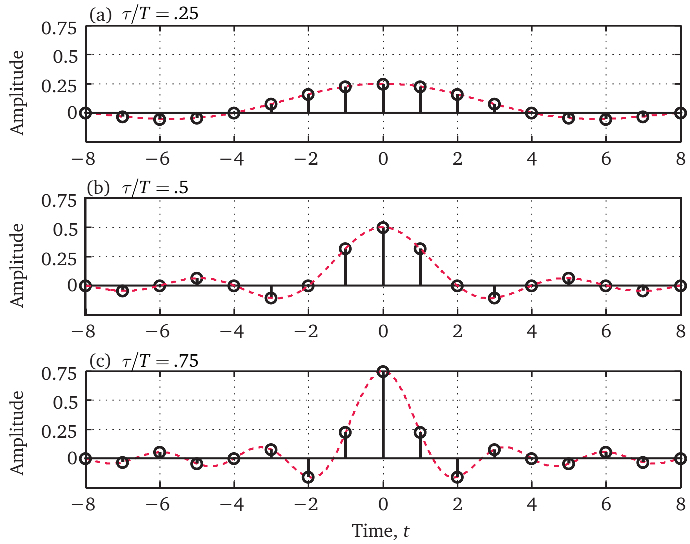

Figure 2.16: Fourier coefficients of a square pulse.
Note that as the pulse gets narrower, its frequency response gets shallower. However, as the duty cycle increases, such as in (c), the response is becoming narrower.

Substituting r in the above equation, we can write the expression for the coefficients of this signal as:

$$C_k = \frac{1}{2}\text{sinc}(k\pi/2) \tag{2.32}$$

This is the *sinc* function. It comes up so often in signal processing that it is probably the second most important equation in DSP after the Euler's equation.

Now, we can plot the coefficients of the repeating continuous-time square pulse coefficients for various duty cycles. Note how the peakedness of the main lobe changes inversely with the duty cycle. A narrow pulse relative to the period in Fig. 2.16(a), has a shallower frequency response than one that takes up more of the period. The zero crossings occur at inverse of the duty cycle. For $r = 0.5$, the zero crossing occurs at $k = 2$, for $r = 0.25$, the crossing is at $k = 4$, and for $r = 0.75$, the crossing occurs at $n = 1.33$. At $r = 1.0$, the pulse would be a flat line and it will have an impulse as its frequency response. For very small r, the pulse is delta function-like and the response will go to a flat line. Note the usage of words, **frequency response**. This is just another name for the *spectrum*.

Although the equation for this function is fairly easy, it takes a little while to develop intuitive feeling. We cannot overemphasize the importance of this signal and you ought to spend some time playing around with the parameters to understand the effect. We will of course keep coming back to it in the next few chapters.

In this chapter, we covered the complex form of the Fourier series as a prelude to the next topic, the Fourier transform. We see that even though the time domain function is continuous and periodic, the Fourier series coefficients (FSC) and hence the spectrum developed is discrete. In Chapter 3, we will see how that affects the Fourier analysis.

Summary of Chapter 2

In this chapter, we looked at the complex exponential as a concise way of representing the Fourier series equation. The complex exponentials make the Fourier series math easier. The spectrum however is now shown as double-sided, which means that the frequency index spans from $-\infty$ to $+\infty$. This has the effect of splitting the trigonometric coefficients into half.
Terms used in this chapter:

- **Euler's equation**
- **Continuous-time complex exponential**, $e^{j\omega t}$ of frequency ω.
- **Complex coefficients of the Fourier series**, C_k
- **Double-sided spectrum**

1. The Euler equation defines a complex exponential as a 3D function consisting of a cosine and sine in quadrature.

$$\begin{aligned} e^{j\omega t} &= \cos\omega t + j\sin\omega t \\ e^{-j\omega t} &= \cos\omega t - j\sin\omega t \end{aligned}$$

2. The Euler expression of a CE can have a positive or a negative exponent. The sign change indicates a change of direction of the function.
3. The sine and cosine alternately can be represented by two CE of different signs. We use the following expressions to represent them in the Fourier series to obtain a complex form of the Fourier series equation.

$$\begin{aligned} \cos(k\omega t) &= \frac{1}{2}\left(e^{jk\omega t} + e^{-jk\omega t}\right) \\ \sin(k\omega t) &= \frac{1}{2j}\left(e^{jk\omega t} - e^{-jk\omega t}\right) \end{aligned}$$

4. To represent a periodic signal using the complex exponentials requires a double-sided harmonic index k, unlike the trigonometric case where the harmonic index is positive.
5. The harmonic index extends from $-K \le k \le +K$. K can span from -∞ to $+\infty$.
6. The x-axis now represents values from $-K\omega_0 \le k\omega_0 \le +K\omega_0$ and this is often read as representing negative frequency when, in fact, it is the index that is negative.

7. The FSC instead of being of three types can now be represented by a single equation:

$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t) e^{-jk\omega_0 t} \, dt$$

8. The fundamental properties remain the same, the time in this representation is continuous and frequency is discrete with index k, which is an integer.
9. The amplitude spectrum obtained from the complex representation looks different from the one-sided spectrum. We call this spectrum a two-sided spectrum. The amplitude value for a particular harmonic is now split in half for the positive index (so-called positive frequency) and half for the negative index (so-called negative frequency). The 0th component, however, remains the same. This conserves energy and makes both forms equivalent.

Questions

1. What is a complex exponential?
2. What is the expression for a sine in the complex form?
3. If we look along this axis, the CE appears as a circle. What axis is that? Which parameter is not visible in this view?
4. What is the value of $e^{-j\pi}$, $e^{j2\pi}$, and $e^{-j\frac{3}{4}\pi}$.
5. Why is this equation true? $e^{j\pi}+1=0$
6. What is the difference between two complex exponential of the same exponent but different signs, such as $e^{j\omega_k t}$ and $e^{-j\omega_k t}$. If we add these two signals, what do we get in the complex domain?
7. What dimensional space is required to plot a complex exponential?
8. The term phasor is often used in relation to complex exponentials, what is it?
9. If you plot a sinusoid plus its shifted version, $\sin(2\omega t)+\sin(2\omega t+\phi_0)$, what is the phase of the new signal?
10. Represent this sinusoid as a complex exponential: $\cos\left(\omega t+\frac{\pi}{4}\right)$.
11. What is the relationship of the Taylor series to a sinusoid?
12. What is the advantage of using complex exponential as a basis set instead of sines and cosines?
13. Given these CEs, give their expression in the Euler form: e^{-j3}, $(e^{-j4}+e^{-j2})$, and $e^{-j\frac{\pi}{2}t}$.
14. How would you express phase in the complex exponential form?
15. Write the CE form of these signals:
    (a) $\sin(7\pi t+\frac{\pi}{4})$
    (b) $\cos(7\pi t+\frac{\pi}{4})-\sin(7\pi t-\frac{\pi}{4})$
    (c) $\cos(3\pi t+\frac{\pi}{2})$
16. What is the magnitude of this complex signal, e^{-5jt}?
17. The real part of a CE has a peak amplitude of 1 and the imaginary part has the peak amplitude of 2. What is the peak and the average power of this signal?
18. How do we transmit a complex signal? What does it look like?
19. What does division by j mean?
20. What does multiplication by j mean?
21. What is the magnitude and phase of these signals:
    (a) $\left(\sin(\omega t)-\cos(\omega t)\right)$
    (b) $\left(\frac{1}{2}\cos(\omega t)-\sin(\omega t)\right)$
    (c) $\left(2\cos(\omega t)-\frac{1}{2}\sin(\omega t)\right)$
22. What is a single-sided spectrum? What does it represent?
23. Given the amplitude spectrum, how would you compute the power spectrum?
24. What is two-sided spectrum of these signals?
    (a) $f(t)=-\cos(2\pi t)+\cos(9\pi t)+\sin(12\pi t)$
    (b) $f(t)=2\cos(9\pi t)+2\cos(18\pi t)$

25. When plotting a two-sided spectrum, what does the x axis represent?
26. If you are given the real and imaginary components of a signal, how do you compute the phase? Is phase changing with time or frequency?
27. For a complex signal, both real and imaginary signals can have nonzero phase, so what is the phase of a complex signal? How is it different from the phases of the components?
28. What is the relationship of the trigonometric coefficients to the complex coefficients?
29. How do explain the idea of negative frequency?
30. Why the spectrum of a complex signal is always one-sided?
31. What is the mean and the peak power of a sinusoid of amplitude 1.0?
32. If we have a periodic signal of square pulses with a duty cycle of 0.1. How much wider is its spectrum as compared to a pulse that has a duty cycle of 0.5?
33. What happens to the magnitude spectrum if phase of the signal changes?

Appendix A: A little bit about complex numbers

We can use complex numbers to denote quantities that have more than one parameter associated with them. A point in a plane is one example. It has a y coordinate and a x coordinate. Another example is a sine wave: it has a frequency and a phase. The two parts of a complex number are denoted by the terms Real and Imaginary, but the Imaginary part is just as real as the Real part. Both are equally important because they are needed to nail down a physical signal.

The signals traveling through air are real signals and it is only the processing that is done in the complex domain. There is a very real analogy that will make this clear. When you hear a sound, the processing is done by our brains with two orthogonally placed receivers, the ears. The ears hear the sound with slightly different phase and time delay. The received signal by the two ears is different and from this our brains can derive fair amount of information about the direction, amplitude and frequency of the sound. So although yes, most signals are real, the processing is often done in complex plane if we are to drive maximum information.

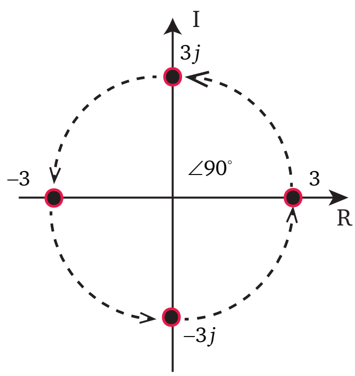

Figure A.1: Multiplication by j shifts the location $0f$ a point on a plane by $90°$.

The concept of complex numbers starts with real numbers as a point on a line. Multiplication of a number by -1 rotates that point $180°$ about the origin on the number line. If a point is 3, then multiplication by -1 makes it -3 and it is now located $180°$ from $+3$ on the number line. Multiplication by -1 can be seen as $180°$ shift. Multiplying this rotated number again by -1, gives the original number back, which is to say by adding another $180°$ shift. Therefore, multiplication by $(-1)^2$ results in a $360°$ shift. What do we have to do to shift a number off the line, say by $90°$? This is where j comes in. Multiply 3 by j, so it becomes $3j$. Where do we plot it now? Herein lies our answer to what multiplication with j does. Multiplication by j moves the point off the line.

Question: What does division by j mean?

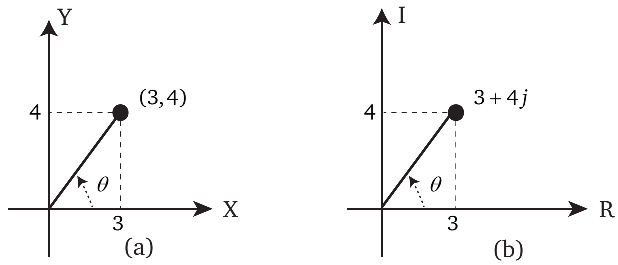

Figure A.2: a. A point is space on a Cartesian diagram. b. Plotting a complex function on a complex plane.

Answer: It is same as multiplying by $-j$.

$$\frac{x}{j} \times \frac{j}{j} = \frac{jx}{-1} = -jx.$$

This is essentially the concept of complex numbers. Complex numbers often perceived as "complicated numbers" follow all the common rules of mathematics. Perhaps, a better name for complex numbers would have been 2D numbers. To further complicate matters, the axes, which were called X and Y in Cartesian mathematics are now called *Real* and *Imaginary*, respectively. Why so? Is the quantity $j3$ any less *real* than 3? This semantic confusion is the unfortunate result of the naming convention of complex numbers and helps to make them confusing, complicated and, of course, complex.

Now let us compare how a number is represented in the complex plane. Plot a complex number, $3 + j4$. In a Cartesian plot, we have the usual $X - Y$ axes and we write this number as $(3, 4)$ indicating 3 units on the X-axis and 4 units in the Y-axis. We can represent this number in a complex plane in two ways. One form is called the rectangular form and is given as:

$$z = x + jy$$

The part with the j is called the imaginary part (although of course it is a real number) and the one without is called the real part. Here 3 is the Real part of z and 4 is the Imaginary part. Both are real numbers of course. Note that when we refer to the imaginary part, we do not include j. The symbol j is there to remind you that this part (the imaginary part) lies on a different axis. Hence writing these as: $Re(z) = x$ and $Im(z) = y$ Alternate form of a complex number is the **polar form**: $z = M\angle\theta$

where M is the magnitude and θ its angle with the real axis. The polar form that looks like a vector and in essence it is, is called a ***Phasor*** in signal processing. This idea comes from circuit analysis. We also use it in signal processing but it seems to cause some conceptual difficulty. Mainly because, unlike in circuit analysis, in signal processing time is important. We are interested in signals in time domain and the phasor, which is a time-independent concept, is confusing.

Question: If $z = Ae^{j\omega t}$ then what is its rectangular form?
Answer: $z = A\cos\omega t + jA\sin\omega t$.

We just substituted the Euler's equation for the complex exponential $e^{j\omega t}$. Think of $e^{j\omega t}$ as a shorthand functional notation for the expression $\cos\omega t + j\sin\omega t$. The real and imaginary parts of z are given by

$$Re(z) = A\cos\omega t \text{ and } Im(z) = A\sin\omega t.$$

Converting forms

1. Given a rectangular form $z = x + jy$ then its polar form is equal to

$$M\angle\theta = \begin{cases} \sqrt{x^2+y^2}\angle\tan^{-1}\frac{y}{x} & \text{if } x \geq 0 \\ \sqrt{x^2+y^2}\angle\left(tan^{-1}\frac{y}{x}+\pi\right) & \text{if } x < 0 \end{cases}$$

2. Given a polar form $M\angle\theta$ then its rectangular form is given by

$$x + jy = M\cos\theta + jM\sin\theta$$

Example 2.11. Convert $z = 5\angle 0.927$ to rectangular form

$$\begin{aligned} Re(z) &= 5\cos(0.927) = 3 \\ Im(z) &= 5\sin(0.927) = 4 \\ \Rightarrow z &= 3 + j4 \end{aligned}$$

Example 2.12. Convert $z = -1 - j$ to polar form

$$\begin{aligned} M &= \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} \\ \theta &= \arctan\frac{y}{x} + \pi, \quad \text{since } x < 0 \\ &= \arctan\frac{-1}{-1} + \pi = \frac{3}{4}\pi \\ \Rightarrow z &= \sqrt{2}\angle\frac{3}{4}\pi. \end{aligned}$$

Example 2.13. Convert $z = 1 + j$ to polar form.

$$\begin{aligned} M &= \sqrt{(1)^2 + (1)^2} = \sqrt{2} \\ \theta &= \arctan\frac{y}{x}, \quad \text{since } x > 0 \\ &= \arctan\frac{1}{1} = \frac{1}{4}\pi \\ \Rightarrow z &= \sqrt{2}\angle\pi/4. \end{aligned}$$

Adding and Multiplying

Add in rectangular form, multiply in polar. Its easier this way.

1. Given $z_1 = a + jb$ and $z_2 = c + jd$ then $z_1 + z_2 = (a + c) + j(b + d)$.
2. Given $z_1 = M_1\angle\theta_1$ and $z_2 = M_2\angle\theta_2$, then $z_1 \cdot z_2 = M_1M_2\angle(\theta_1 + \theta_2)$.

Example 2.14. Add $z_2 = \sqrt{2}\angle 0.785$ and $z_2 = 5\angle 0.927$.

Convert both to rectangular form.

$$z_1 = 1 + j \text{ and } z_2 = 3 + j4$$
$$\Rightarrow z_3 = (1 + 3) + j(1 + 4) = 4 + j5.$$

Example 2.15. Multiply $z_1 = 1 + j$ and $z_2 = 3 + j4$.

First convert to polar form and then multiply. Although multiplying these two complex numbers in rectangular format looks easy, in general that is not the case. Polar form is better for multiplication and division.

$$z_1 \cdot z_2 = \sqrt{2}\angle 0.785 \times 5\angle 0.927 = 5\sqrt{2}\angle 1.71$$

Example 2.16. Divide $z_1 = 1 + j$ and $z_2 = 3 + j4$.

$$\frac{z_1}{z_2} = \frac{\sqrt{2}\angle 0.785}{5\angle 0.927} = \frac{5}{\sqrt{2}}\angle -0.142$$

Conjugation

The conjugate for a complex number z, is given by $z^* = x - jy$. For a complex exponential $e^{j\omega t}$ is the complex conjugate of $e^{-j\omega t}$. In polar format, the complex conjugate is same phasor but rotating in the opposite direction.

Rule: If $z = M\angle\theta$, then $z^* = M\angle -\theta$.

Useful properties of complex conjugates

$$|z|^2 = zz^*$$

This relationship is used to compute the power of the signal. The magnitude of the signal can be computed by half the sum of the signal and its complex conjugate. Note, the imaginary part cancels out in this sum.

$$|z| = \frac{1}{2}\big(z + z^*\big).$$