5

Discrete-time Fourier transform (DTFT) of aperiodic and periodic signals

We started with Fourier series which can represent a periodic signal using sinusoids. Fourier Transform, an extension of the Fourier series was developed specifically for *aperiodic* signals. In chapter 4 we discussed the Fourier transform as applied to *continuous-time signals*. Now we examine the application of Fourier transform to *discrete-time signals*. We already discussed the discrete-time Fourier series (DTFS) as applied to periodic signals in Chapter 3. Here the same ideas are applied to aperiodic signals to obtain the Fourier transform.

Whether periodic or non-periodic, discrete-time signals are the main-stay of signal processing. Signals are collected and processed via sampling, or by devices which are inherently discrete. Despite the fact that sampled signals “look” like their analog parents, there are some major conceptual differences between discrete and continuous signals. However, the fundamental concepts of Fourier transform discussed in Chapter 4 for continuous-time signals apply equally well to discrete-time signals with only minor differences in scaling.

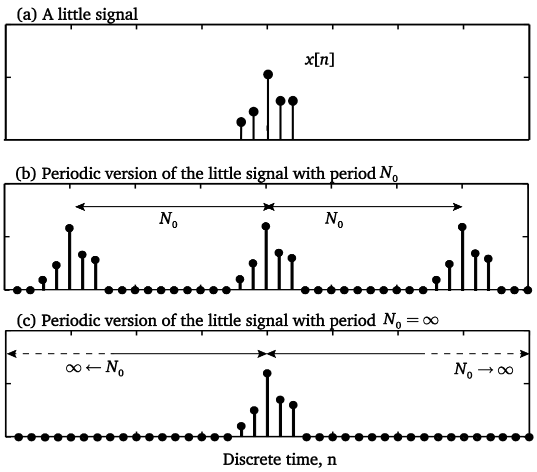


Figure 5.1 – An aperiodic discrete-time signal can be considered periodic if period is assumed to be infinitely long.

Let's do the same thought experiment we did for continuous signals. Given a piece of a discrete and ostensibly aperiodic signal such as in Fig. 5.1, we conceptually extend its period. The signal $x[n]$ is just 5 samples, but we pretend that the signal is periodic with period N_0. But then we say that this period can be very long, maybe even infinitely long. So if we extend the period of this signal to ∞, we basically get back the original signal $x[n]$ which is now surrounded by a sea of zeros.

$$\lim_{N_0 \to \infty} x_{N_0}[n] = x[n] \qquad (5.1)$$

As we increase N_0, in limit the result is the starting signal, but it can now be considered a periodic signal, although only in a mathematical sense. We can't see any of the periods. They are too far apart. And now since the signal is *periodic*, we can use the discrete-time Fourier series (DTFS) to write its frequency representation in terms of complex coefficients as

$$C_k = \lim_{N_0 \to \infty} \frac{1}{N_0} \sum_{n=0}^{N_0-1} x_{N_0}[n] e^{-jk\Omega_0 n} \qquad (5.2)$$

Discrete-time Fourier Transform (DTFT)

Recall that in Chapter 3 we defined the fundamental digital frequency of a discrete periodic signal as $\Omega_0 = \frac{2\pi}{N_0}$, with N_0 as the period of the signal in samples. As N_0 goes to infinity, from this definition, the fundamental frequency goes to zeros as well. Hence the harmonics which are defined as integer multiple of the fundamental frequency, such as $k \times \Omega_0$ also lose meaning. We can also think of the fundamental frequency as the resolution of the frequency response, so if this number is zero, then the frequency becomes continuous and *k*, the harmonic identifier drops out entirely. Also the signal itself from Eq. (5.1) can be written as a periodic signal x[n], dropping all the limits etc.. Just as we discussed in Chapter 4, the result of extending the period to infinity leads to a frequency response which is continuous in frequency, even though the signal itself is discrete in this case.

Now we define a new transform called the ***Discrete-time Fourier Transform*** of an *aperiodic* signal as

$$\text{DTFT} \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \qquad (5.3)$$

Here $x[n]$ is an *aperiodic discrete-time* signal. In Chapter 4 we defined the continuous-time Fourier transform as given by

$$\text{CTFT} \qquad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad (5.4)$$

Notice the similarity between these two transforms. The CTFT $X(\omega)$, of the continuous-time signal x(t) is also continuous in frequency. The DTFT or $X(\Omega)$ is continuous in frequency for the same reason that the CTFT is continuous: due to the extension of the period to ∞. Both $X(\omega)$ and $X(\Omega)$ are shown with round brackets for this reason.

The CTFT frequency is termed ω whereas the digital frequency in DTFT is given this symbol, Ω. The digital frequency Ω as we learned in Chapter 3 is unique only over a single 2π range. The DTFT, same as the CTFT, is a way of expressing the signal x[n] using harmonic exponentials. The spectrum in each case are the relative magnitudes of these harmonics.

The continuous and infinitely long functions are great in textbooks but impractical in real life. So a *continuous time spectrum* is actually not a desirable result. What we want is a *discrete* response, which is far more practical. Discrete data is much easier to store, manage and manipulate. As engineers working with numbers, we want a spectrum that is discrete and one which we can compute in a discrete manner using computers. However we are not quite there yet.

Starting from the definition of DTFT in Eq. (5.3), we now derive the time domain function, $x[n]$ that resulted in this $\mathrm{X}(\Omega)$ or the DTFT. We refer to this as "taking the inverse DTFT". This is written in short-hand as iDTFT. To take the iDTFT, we multiply both sides of Eq. (5.3) by $e^{j\Omega n}/2\pi$ and then integrate both sides over 2π. In this case m is the dummy variable.

Commented [Ma1]:

$$X(\Omega)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\, d\Omega = \tag{5.5}$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} x[n]e^{-j\Omega m}\right)e^{j\Omega n}\, d\Omega$$

Now we continue to manipulate the RHS.

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\underline{\left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega m}\right)}e^{j\Omega n}\, d\Omega$$

$$=\sum_{m=-\infty}^{\infty} x[m]\underline{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(\mathrm{m}-n)}d\Omega\right)}$$

(5.6)

The last row is still complicated looking. But we note that the underlined part in the last row is summation of the complex exponentials (CE) $e^{j\Omega(\mathrm{m}-n)}$ and is in fact equal to shifted delta function, $\delta(\mathrm{m}-n)$. Now normally in most books, this would be left as "an exercise for the student", but we take you on a detour to examine this point. However, instead of showing that integration of $e^{j\Omega(\mathrm{m}-n)}$ is equivalent to a shifted impulse, we will show only that integration of $e^{j\Omega(n)}$ is equal to a single impulse. From there we ask you to extrapolate the result. In fact a sinc function in discrete time looks just like a delta function and is in fact equivalent to a delta function.

A Sinc detour

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega k} d\Omega &= \frac{1}{j2\pi k} e^{j\Omega k}\Big|_{-\pi}^{\pi} \\
&= \frac{1}{j2\pi k}\left[e^{j\pi k} - e^{-j\pi k}\right] \\
&= \frac{1}{2j}\left[e^{j\pi k} - e^{-j\pi k}\right]\frac{1}{\pi k} = \frac{1}{2j}\ \cos(\pi k) + j\sin(\pi k) - \cos(\pi k) + j\sin(\pi k) \\
&= \frac{\sin(\pi k)}{\pi k} \\
&= \operatorname{sinc}\ k\ = \delta(k) \qquad \textit{for all } k = \text{integer}
\end{aligned}
$$

(5.7)

The sinc function is 0 for integer values of k, except at $k = 0$, when its value is 1 as we see in Fig. 5.2. Hence the sinc function can be equated to a delta function for the discrete case. We generalize this finding of the sinc function to the following shifted case.

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(k-n)} d\Omega = \delta(k-n) \qquad (5.8)
$$

The second expression says that the summation of time shifted CEs gives us a time shifted delta function in discrete-time.

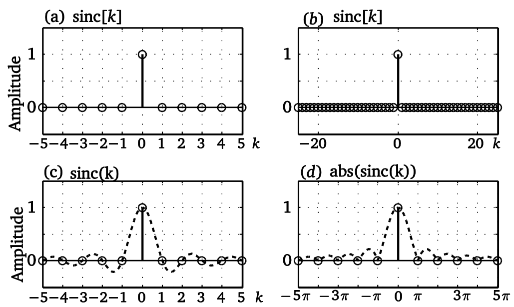


Figure 5.2 - (a) The discrete sinc, (b) longer version of sinc still looks like a delta function, (c) the continuous version of the sinc and the discrete values, and (d) the magnitude of the sinc function.

F5_ sinc

Fig. 5.2(a) shows the discrete sinc function. For every value of k, an integer other than 0, the sinc function is equal to 0. It looks suspiciously like a delta function here. As we increase k, we see in (b) it keeps looking like a delta function. This equivalence to a delta function is in fact not a function of the length. In (c) we plot the continuous values of the sinc function along with the discrete values. For values of k, not integers, we now get non-zero values which shows us what the function looks like in continuous-time. The delta function "look" of the sinc function is a form of discrete deception. The discrete version is picking up only the values at certain points which are all zero. None others are computed nor can we see them. So for all practical purposes, a discrete version of the sinc is a delta function. In (d), we see the same continuous functions as in (c) but with its absolute value, as it is typically shown in books. Also note that the sinc function is crossing the zero axis at integer values of π. So we can write the x axis in terms π as shown in (d).

Back to the DTFT

Substituting Eq. (5.8) into Eq. (5.6), we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega) e^{j\Omega m} d\Omega = \sum_{n=-\infty}^{\infty} x[\mathrm{n}]\delta(\mathrm{n}-m)$$

From the sifting property of the delta function, the right hand side becomes $x[\mathrm{m}]$, and since this not a function of n, we end here. Hence the inverse DTFT of Eq. (5.3) is the time domain function $x[n]$.

iDTFT	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$	(5.9)
DTFT	$X(\Omega) = \sum_{k=-\infty}^{\infty} x[n] e^{-j\Omega k}$	

The forward transform or the DTFT is denoted by symbol $X(\Omega)$. However, you will find other ways of denoting the DTFT. Oppenheimer book refers to it as $X(e^{j\Omega})$, whereas both Miral book and the Lathi and Green books refers to it by $X(\Omega)$. These notations are basically convention and not that important. In speaking, most all of these forms are referred to as simply as the "Fourier Transform" or even the more generic "spectrum". And even more egregiously some people call this a FFT, which it may or may not be. Since most signals we deal with in practice are discrete, the time qualifier is dropped and we can just call these as the Fourier transform. However, in this book we will continue to refer to each type of transform by its full formal name, CTFT, DTFT, DFT etc.

Magnitude and phase spectrums

The DTFT $X(\Omega)$ is generally a complex function. We can show it in two ways, just as all the other Fourier representations, either as its real and imaginary components which are the coefficients of the cosine and sine harmonics or we can show them by magnitude and phase. Both methods are often employed as needed.

$$X(\Omega) = \underbrace{|X(\Omega)|}_{Magnitude} \underbrace{e^{j\angle X(\Omega)}}_{Phase}$$

For real signals, the magnitude is symmetrical (or even) and the phase is odd.

DTFT is continuous and periodic with period of 2π

So first we say that the period of a signal is assumed to be infinitely long and now we are saying that the DTFT is periodic. How can that be? Well what we mean by that is the spectrum i.e. $X(\Omega)$ repeats with 2π. This talk of a frequency that is measured in π's can be very confusing. But we must accept the fact that the DTFT is defined in terms of the *digital frequency* and not Hz. The signal consists of discrete values and hence can only be represented by N harmonics, the length of the signal and nothing else. In order to make the analysis independent of real time, i.e. the time between the samples, DTFT is instead defined in terms of radial movement. This, if you trust us, makes the math, easier (ha!).

Unlike continuous frequency, discrete frequency, Ω, with units of radians per sample, lacks a time dimension. It is periodic only in an angular sense. It is unique for values in only one range. We should not think of it as number of samples or, or *time* it takes to cover 2π radians because that is just not part of its definition. The range $-\pi \le 0 \le \pi$ of the digital frequency and the spectrum computed thereof is called the *principal alias* as we noted in Chapter 3.

Because of this condition, the coefficients for harmonic frequencies outside 0 to 2π are just copies. Hence there is no need to compute $X(\Omega)$ outside the 2π range. Anything beyond that just repeats the same values from the 2π range, or in fact from any such range. We can ignore all these “replicated spectrums” as they are identical to the principal alias. We write this property as

$$X(\Omega) = X(\Omega + 2m\pi) \qquad \text{for all } \Omega \text{ in range of } (-\pi, \pi),\ m \text{ an integer} \qquad (5.10)$$

This comes from the simple observation that

$$X(\Omega+2m\pi)=\sum_{n=-\infty}^{\infty}x[n]e^{j(\Omega+2\pi m)n}$$
$$=\sum_{n=-\infty}^{\infty}x[n]e^{j\Omega n}\underbrace{e^{-j2\pi nm}}_{=1}=X(\Omega)$$

Every 2π , $X(\Omega)$ is identical to the one before. This property simplifies the computation as we need only integrate over a 2π range of the digital frequency. Since the area under a periodic signal for one period does not change no matter where you start the integration, we can generalize the DTFT equation over any range. We can for example write the equation for the iDTFT in the second manner, with integration range written as just 2π , and both are valid.

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\Omega)e^{j\Omega n}d\Omega$$
$$=\frac{1}{2\pi}\int_{2\pi}X(\Omega)e^{j\Omega n}d\Omega$$

Comparing CTFT with DTFT

Let's examine how CTFT compares to the DTFT for an *aperiodic* signal.

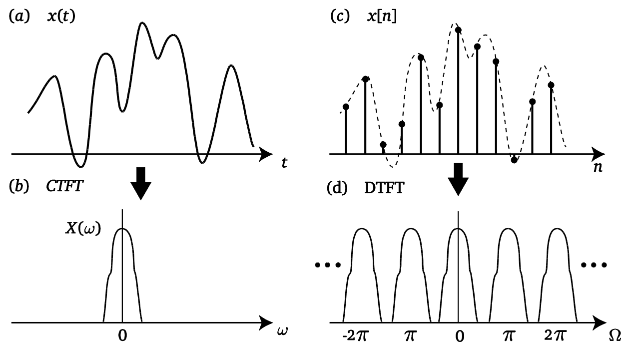


Figure 5.3 - Comparing CTFT with DTFT
(a) aperiodic CT signal, (b) its CTFT is continuous, (c) a sampled discrete signal (d) is same as (b) but repeats with 2π.

Both CTFT and DTFT have a very similar construct. Assume that the CTFT of the signal shown in Fig. 5.3(a) is as shown in (b). We see a single spectral mass around zero frequency with a continuous frequency resolution. Now take the same signal in discrete form (c) and it has a DTFT that is continuous just as the CTFT, but this one repeats with 2π radians. This is the same result we showed for discrete-time Fourier series. So note that DTFS and DTFT are very similar.

We will now show some examples of the DTFT. In these examples, we compute only the principal alias which is the DTFT around the zero frequency, from $-\pi$ *to* $+\pi$. However, we must not lose sight of the fact that the DTFT spectrum copies go on forever on each side of the principal alias, as we see conceptually in Fig. 5.3(d).

The DTFT has the same behavior as the CTFT in most cases when the signal is bandlimited to 2π. The CTFT properties shown in Chapter 4, Table I are equally valid in conceptual sense for discrete signals. These properties can be used to compute the DTFT for many signals, starting with the knowledge of the DTFT of some of the basic signals. We can in most cases take a CTFT equation, change the continuous frequency ω to digital frequency notation Ω and then change continuous time *t* to discrete time notation *n* and get a valid expression for the DTFT. However, what we get this way is only the principal alias because CTFT does not repeat. We must recognize that DTFT repeats forever.

The DTFT is a bridge topic to get us to the ***Discrete Fourier transform (DFT)***, a widely employed and a very useful algorithm. DFT is *discrete* in both time and frequency domain and can be calculated easily by software such as Matlab. The ***Fast Fourier Transform (FFT)*** was developed to make computation of the DFT quick and efficient. It is just a computation algorithm and not a unique type of Fourier transform. The DTFTs for most signals other than a few simple ones you see in text books are hard to compute, requiring one to pull out integral tables. Nor are they commonly used in real-life engineering. So why bother with DTFT if the subject is so theoretical? The main reason is that until we understand DTFT, we cannot fully appreciate the DFT. Ehen learned as a stand-alone topic, the DFT makes sense only in a procedural sense but one lacks deeper understanding of where it is coming from.

DTFT from CTFT

DTFT can be obtained directly from a CTFT. Let’s compute the DTFT of a signal whose CTFT we know.

$$x(t)=1, \quad X(\omega)=2\pi\delta(\omega)$$
$$x[n]=1, \quad X(\Omega)=?$$

This is a trivial case. By making the appropriate changes, we get

$$X(\Omega)=2\pi\delta(\Omega) \quad \mathit{for} -\pi \le \Omega \le \pi$$

This is the principal alias only. The complete DTFT repeats, just as we have repeated ourselves endlessly, so we extend the above expression to

$$X(\Omega)=2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k) \quad \mathit{for\ all}\ \Omega$$

For each k, we get an impulse at frequency $\Omega=2\pi k$, so this says that the spectrum of a constant discrete signal is ever repeating impulses at integer multiples of 2π.

DTFT of a delayed impulse

The delayed impulse $x[n]=\delta[n-n_0]$, is a very important signal. Nearly all discrete signals can be decomposed as a summation of this signal. Again we can take the CTFT of a delayed impulse and change the terms to their discrete equivalents but instead we will do the math for this case using the DTFT equation. We compute the DTFT of a delayed unit-impulse function, $x[n]=\delta[n-n_0]$ using the DTFT Eq. (5.6).

$$X[\Omega]=\sum_{n=-\infty}^{\infty}\delta[n-n_0]\,e^{-j\Omega n}$$

The product of functions, $\delta[n-n_0]$ and $e^{-j\Omega n}$ is non-zero only at point n_0, so we simplify the RHS as

$$X[\Omega]=\sum_{n=-\infty}^{\infty} e^{-j\Omega n_0}$$

Since $e^{-j\Omega n_0}$ is not a function of *n*, we can ignore the summation and the DTFT is simply equal to

$$\begin{aligned} x[n] &= \delta(n-n_0) \\ X[\Omega] &= e^{-j\Omega n_0} \end{aligned} \qquad (5.11)$$

The magnitude of this transform is equal to

$$|X[\Omega]| = |e^{-j\Omega n_0}| = |\cos(\Omega n_0) - j\sin(\Omega n_0)| = 1$$

So matter what the shift, the magnitude remains the same. The phase however is

$$\angle X(\Omega) = a\tan\frac{\sin(\Omega n_0)}{\cos(\Omega n_0)}$$

And will change with the shift. This result is exactly the same as if we had applied the time-shift property to a zero-shift delta function. The time shift property states that the Fourier transform of a shifted signal is equal to the Fourier transform of the un-shifted signal times a CE of frequency n_0, which is exactly what we have here. If the shift is equal to 0, then we get

$$\begin{aligned} x[n] &= \delta(n-0) \\ X[\Omega] &= e^{-j\Omega n_0} = 1 \end{aligned} \tag{5.12}$$

The transform of the un-shifted delta signal is of course 1 as we see in Figure 5.2(b) and we can see that the DTFT of this signal is a purely continuous function of Ω. If n_0 = 2, we get

$$\begin{aligned} x[n] &= \delta(n-2) \\ X[\Omega] &= e^{-j2\Omega} = \cos(2\Omega) - j\sin(2\Omega) \end{aligned}$$

In figure 5.4, we see the effect of the delay on the transform of the delayed function, with no change in magnitude but the phase change by 4π, with 2π phase delay per sample delay. We see a total of 4π phase travel over the range in (f).

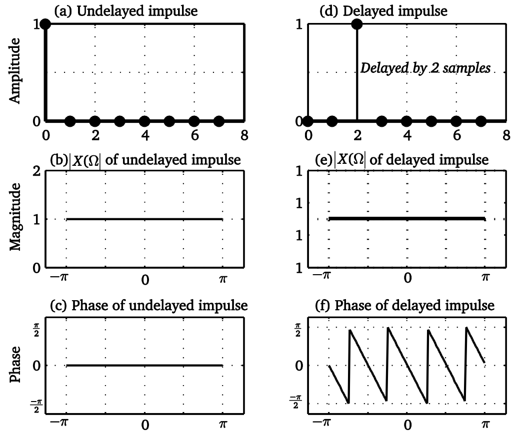


Figure 5.4 - Comparing $X(\Omega)$ of an unshifted and shifted impulse.
F5_54Impulse

Linear superposition of impulses

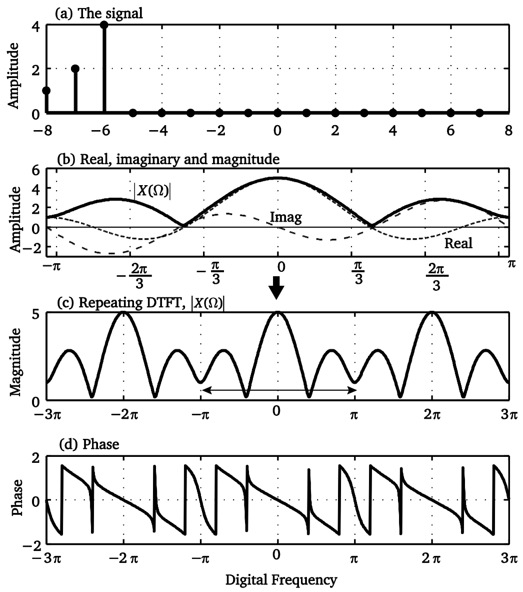
We now compute DTFT of a discrete signal that combines several shifted impulse functions.

$$x[n] = \delta[n] + 2\delta[n-1] + 4\delta[n-2]$$

We treat each one of these delta function individually by applying the linearity principal.

$$\begin{aligned} X(\Omega) &= \sum_{k=-\infty}^{\infty} \delta[n]e^{-j\Omega n} + 2\sum_{k=-\infty}^{\infty} \delta[n-1]e^{-j\Omega n} + 4\sum_{k=-\infty}^{\infty} \delta[n-2]e^{-j\Omega n} \\ &= 1 + 2e^{-j1\Omega} + 4e^{j2\Omega} \\ &= 1 + \cos(\Omega) + j\sin(\Omega) + 2\cos(2\Omega) - 2\sin(2\Omega) \\ &= \underbrace{1 + \cos(\Omega) + 2\cos(2\Omega)}_{real} + \underbrace{j\sin(\Omega) - 2j\sin(2\Omega)}_{\text{Img}} \end{aligned}$$

Note that since the digital frequency Ω, has units of radians, we do not have a time variable to go along with it.



F5_55DTFT of sampleseq

Figure 5.5 – DTFT of (a) the discrete aperiodic signal, (b) Principal alias, (c) its phase and (d) the real repeating DTFT.

We note that the DTFT is continuous and repeats with 2π. The spectrum shown covers 3 periods, the rest are all there, outside the bounderies of the plot. We don't show them but they are indeed there.

Let's take a look at another way of computing the DTFT of the shifted impulses of this signal.

$$x[n] = 2\delta[n-1] - 3\delta[n-2] + 5\delta[n-4] \ .$$

Here instead of doing the math, we will apply the time-shift property to each of these delta functions. The DTFT of a delayed delta function is a CE of frequency, $e^{-j\Omega n_0}$ as per result from DTFT of a delayed impulse. From that, we write the DTFT of this composite function easily, as the summation of the individual terms. Each of the delta functions corresponds in frequency domain to a frequency of the delay as in Eq. (5.11). Once again, note that the units of the digital frequency are radians as the x-axis of the spectrum is usually given in terms of π.

$$x[n] = 2\delta[n-1] - 3\delta[n-2] + 5\delta[n-4]$$

$$X[\Omega] = 2e^{-j\Omega} \quad - 3e^{-j2\Omega} \quad + 5e^{-j4\Omega}$$

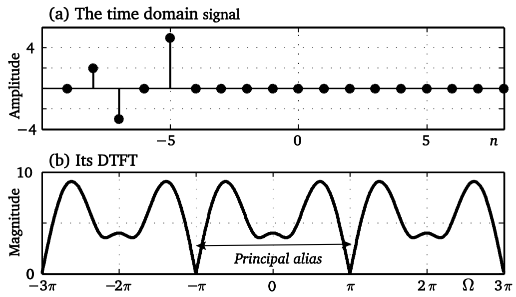


Figure 5.6 - (a) The discrete aperiodic signal, (b) the repeating DTFT, with 2 copies shown.

F5-Ex3

DTFT of a square pulse

Square waves are very useful as a model of real signals. What you learn from these signals, you can then generalize to nearly all shapes. Let's start with a square pulse of width N discrete samples. However, note that signal length is not N, it is longer and in fact is it not infinitely long? Hence the parameter N has nothing to do with the length of the signal. It is just the width of the pulse itself and not the length of the period that matters. Note that in this example, the square pulse is centered at 0. We assume that N is odd. We define this function as

$$x[n] = \begin{cases} 1 & -(N-1)/2 < n \le (N-1)/2 \\ 0 & \textit{Elsewhere} \end{cases}$$

We compute the DTFT as

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega n} = \sum_{n=-(N-1)/2}^{(N-1)/2} 1 e^{-j\Omega n}$$
$$= \frac{\sin\left(\frac{2N+1}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)} \qquad (5.13)$$

The result in the last row is the ***Drichlet Function.*** (Peter Gustav Lejeune Drichlet , a German mathematician)

We can also write the result as follows.

$$X(\Omega) = Diric(N, \Omega) \qquad (5.14)$$

Now we plot the DTFT using Eq. (5.14) for various values of N, which is the width of the square pulse in samples. The absolute value of the Drichlet function is plotted vs. the true value in the RHS of Fig. 5.7. The length of the signal in the LHS is12 samples in each case. Can you say what would happen to the DTFT on the RHS, if we increase the length of the signal from 12 samples to 100 samples. Actually nothing would change, we would get exactly the same function. DTFT is not a function of the total number of samples beyond the pulse. The DTFT as we can see in (5.14) is a function of only the number of samples of the square pulse or N. That's because the formulation of the DTFT already assumes that zeros on the sides go on forever.

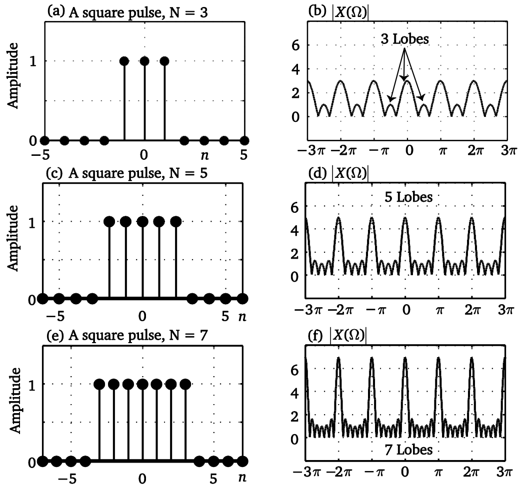


Figure 5.7 – A pulse of length N = 3, 5, 7 samples and its spectrum
F5-squarewaves

Drichlet detour

The Drichlet function in the DTFT of an aperiodic square pulse is often called the periodic version of the *sinc function*. These two are phenomenally important functions in signal processing.

$$\sin c(x) = \frac{\sin(\omega)}{\omega}$$

$$drichlet(x) = \frac{\sin(N\omega/2)}{\mathrm{N}\sin(\frac{1}{2}\omega)}$$

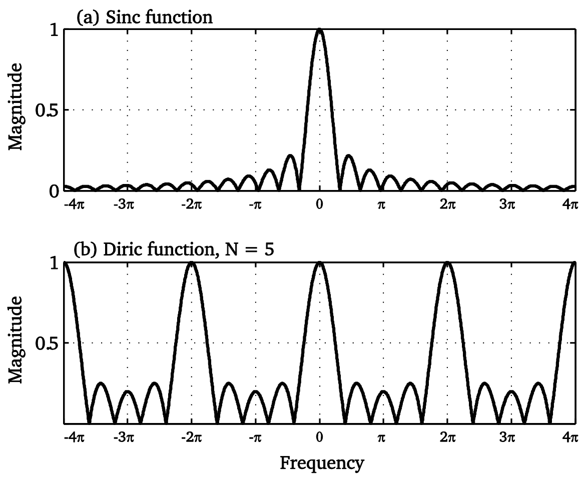


Figure 5.8 - Sinc and the Drichlet function

We plot both of these functions in Fig. 5.8. The Sinc function in the top row is continuous and is aperiodic. The Drichlet in the second row appears similar to the Sinc and but is periodic with 2π, for N = odd and with a period of 4π when N is even. (We don't see this in Fig. 5.8, because the plot contains absolute values so all the lobes are on the positive side.)

Let's examine the Drichlet function in a bit more detail. Fig. 5.9 shows the behavior of the Diric function (the Matlab version of the Drichlet) as a function of digital frequency. Recall that units of digital frequency are radians. We see that the number of zero crossings in the range of 2π, are equal to N – 1. For N = 5, we see 4 zero crossings, for N = 9, we see 8 zero crossing. The function is non-zero only at 0. On RHS, we see the same function over a longer range of digital frequencies plotted along with a sampled-discrete version. The discrete version of this signal is interesting. It looks like an impulse train just as did the sinc function for a single impulse.

Where the sinc function looks like a single impulse when sampled, Drichlet looks like an impulse train, with impulses present every 2π . The discrete version is shown by the dots. The Diric function crosses zeros at all frequencies equal to

$(2\pi m / N)$ where N is the order of the diric function as in Matlab diric(f, N). Hence for N = 5, the zeros occur at $\Omega = \pm 2\pi/5,\ \pm 4\pi/5,\ \pm 6\pi/5,\ \pm 8\pi/5, \cdots$, for N = 6, the zeros occur at $\Omega = \pm 2\pi/6,\ \pm 4\pi/6,\ \pm\pi,\ \pm 8\pi/6, \cdots$ etc.

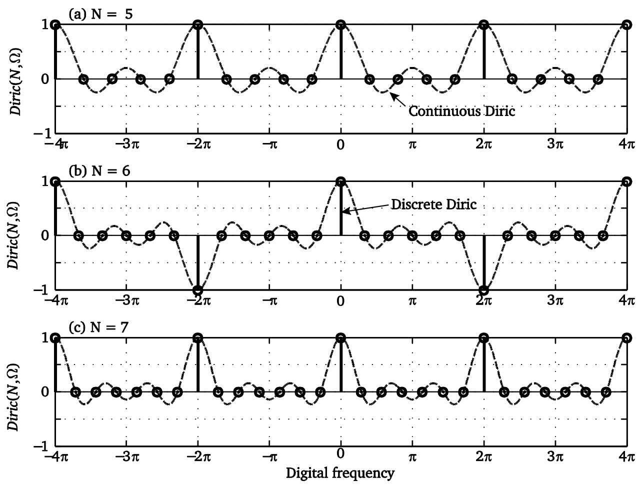


Figure 5.9 - The Drichlet function (a) N = 5, (b) N = 7, (c) N = 9. The x-axis is in radians. The zero crossings occur at $n2\pi / N$.

F5_55Driric for N567

Applying time-shift property to the DTFT of a square pulse

What is the DTFT of a square pulse, when not centered at 0? We can think of this as a square pulse located at zero frequency but with a time-shift. Knowing the time-shift property is a very handy thing. The analysis is same as in un-shifted case, except we are going to add a time shift. We assume that the pulses are centered at L samples from the origin. The time-shift is L units. The DTFT can now be written from the time shift property as simply the DTFT times the CE of frequency per Eq. (5.11) $e^{-j\Omega L}$ as follows

$$X[\Omega] = e^{-j\Omega L} X[\Omega]_{undelayed}$$

$$= e^{-j\Omega L} \frac{\sin\left(\frac{2N+1}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)} \tag{5.15}$$

In Fig. 5.8, we plot DTFT for two pulse widths, N= 3 and, N = 5. Each with shift, L = 10 samples. On the LHS, we see the un-shifted square pulse, on the RHS, the shifted version. We see from Eq. (5.15) that the magnitude of the DTFT did not change, the shift results in no change in the magnitude of the DTFT, only the phase. We see the same thing for N = 5 in Fig. 5.9

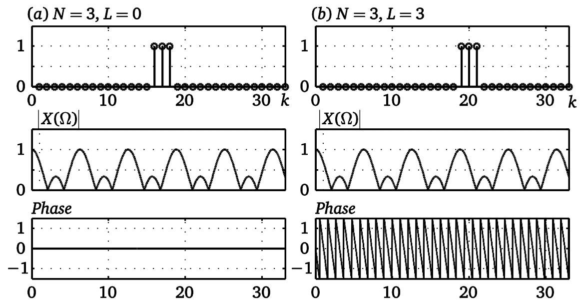


Figure 5.8 – Time shift property for N = 3

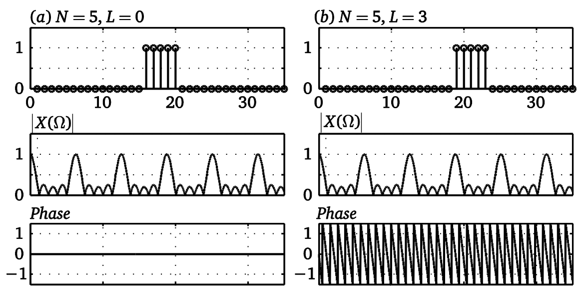


Figure 5.9 - Time Shift property for N = 5

For both cases and in fact all cases, a time shift shifts the phase but the magnitude stays the same.

Time expansion property

Let's take this signal which looks like it has zeros inserted in a 3 sample square pulse, $x = [1001001]$. We can write this discrete signal as

$$x = \delta(\mathrm{n}) + \delta(n-3) + \delta(n-6)$$

There are many different ways of computing the DTFT of such a signal. Here we apply the time-expansion property to show a more efficient method. The pulse of N = 3 has been expanded by a factor of 3 by inserting these zeros. We write the time-expansion property as

$$x(at) \xleftrightarrow{\Im} \frac{1}{|a|} X\left(\frac{\Omega}{a}\right) \qquad (5.16)$$

The DTFT of a square pulse for N = 3, when interpolated with two zeros shrinks by a factor of 3. We can of course compute the DTFT directly as we did for the shifted impulse case. This is equal to

$$X(\Omega) = \frac{1}{3} \frac{\sin \frac{\Omega}{3} \; N + \frac{1}{2}}{\sin \frac{1}{6}\Omega} \qquad (5.17)$$

We see that adding zeros between the samples, expands the signal but compresses the DTFT. In Fig. 5.9, we see this effect as more zeros are added. What happens if N goes to infinity? Then only one delta function is left, and the DTFT will turn into a flat line.

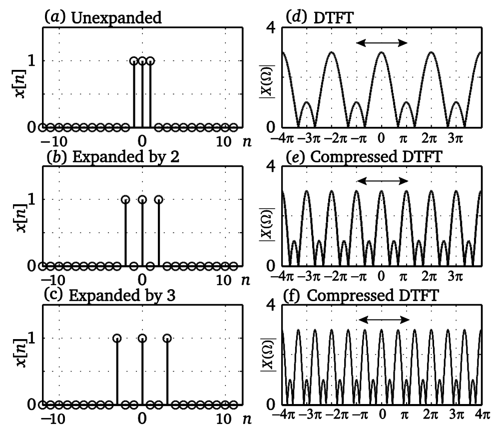


Figure 5.10 - Time expansion property

DTFT of a triangular-shaped pulse

A triangular pulse is nearly as important in signal processing as the square pulse. It is the convolution of two rectangular pulses, something which comes up often. We write the triangular pulse as

$$x[n] = 1 - \frac{|n|}{N} \qquad |n| < N$$

The pulse is 2N samples wide and symmetrical. The DTFT is computed as

$$\begin{aligned} X[\Omega] &= \sum_{n=-\infty}^{\infty} x[n] e^{j\Omega n} \\ &= 1 + \sum_{n=1}^{N-1} \left(1 - \frac{n}{N}\right) e^{j\Omega n} + e^{-j\Omega n} \\ &= 1 + 2\sum_{n=1}^{N-1} \left(1 - \frac{n}{N}\right) \cos(n\Omega) \\ &= \frac{\sin^2(N\Omega/2)}{N\sin^2(\Omega/2)} \end{aligned} \tag{5.18}$$

The result is a function that is the Drichlet function squared.

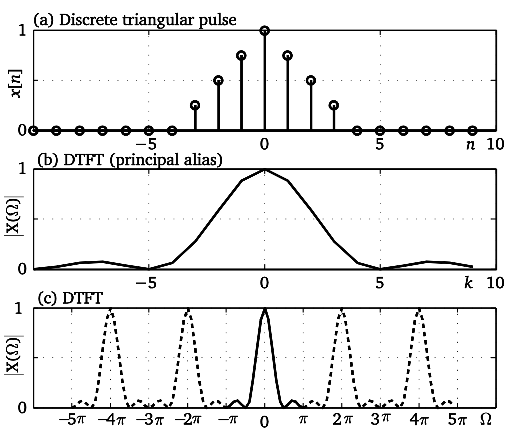


Figure 5.11 - A triangular shape pulse has a squared Diric signal response.

We could have also computed the DTFT of a triangular function by applying the convolution property. We recognize, that a triangle pulse is the result of a convolution of two identical rectangles. So we write the pulse as a convolution.

$$x[n] = rect\left[\frac{n}{N}\right] * rect\left[\frac{n}{N}\right] \qquad (5.19)$$

The DTFT of this convolution is the product of the DTFT of the individual square pulses. From that we get

$$X[\Omega] = \left[\frac{\sin\left(\frac{2N+1}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}\right]^2 \qquad (5.20)$$

So knowing the properties can make the task of computing FTs easier in many cases.

Computing the DTFT of a Raised cosine pulse

The ubiquitous raised cosine pulses are used to transmit communications signals. They limit the bandwidth of the baseband signal and are easily built in hardware.

$$p[n] = \frac{\cos\left(\pi\alpha \frac{n/F_s}{T_s}\right)}{1-\left(2\alpha \frac{n/F_s}{T_s}\right)} \times \frac{\sin\left(\pi \frac{n/F_s}{T_s}\right)}{\pi \frac{n/F_s}{T_s}} \qquad (5.21)$$

Here F_s is the sampling frequency, α is a real number less than one and is called the roll-off factor, T_s is the inverse of symbol rate R_s . The first part is called the raised cosine and the second part which is the sinc function is called the *cascaded sinc* applied to the raised cosine pulse. If alpha = 0, we get an ideal rectangular shape, and if alpha = 1, we get a pure raised cosine shape. These parameters set the baseband bandwidth of the signal as

$$BW = R_s(1+\alpha) \qquad (5.22)$$

To compute the DTFT of this pulse we will have to resort to some heavy-duty math. But no need, as it has already been done for us by better minds. Here is the equation that gives us the CTFT of the above good looking pulse.

$$P(f)=\begin{cases} T_s & 0 \le |f| \le \dfrac{1-\alpha}{2T_s} \\ \dfrac{T_s}{2}\left\{1+\cos\left[\dfrac{\pi T_s}{\alpha}\left(|f|-\dfrac{1-\alpha}{2T_s}\right)\right]\right\} & \dfrac{1-\alpha}{2T_s} \le |f| \le \dfrac{1+\alpha}{2T_s} \\ 0 & |f| \ge \dfrac{1+\alpha}{2T_s} \end{cases} \quad (5.23)$$

We plot the time-domain signal and its DTFT in Fig. 5.11. It looks very similar to a sinc function. Although this pulse goes on forever, for practical design, it is clipped to a certain length.

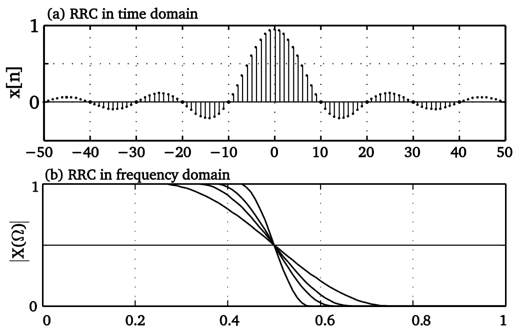


Figure 5.12 - (a) Time domain root-raised cosine pulse shape (b) The spectrum of the raised cosine pulse for $\alpha = .5, .33, .25, .15$
Note that frequency domain looks like a low pass filter.

DTFT of a Gaussian pulse

The discrete-time version of the Gaussian signal is given by

$$x[n]=\frac{1}{\sigma\sqrt{2\pi}}e^{-n^2/2\sigma^2} \qquad (5.24)$$

To compute the DTFT of this signal, being good engineers we are going to again skip the math. Others have already done it for us. The DTFT of this Gaussian shaped pulse, is also Gaussian in shape. You recognize why that happens; because the pulse is an exponential and the integral of such a function is also an exponential. The result is beautiful and elegant and a very useful thing to know. Many random signals are Gaussian in nature. Life is Gaussian and it is Gaussian in all its dimensions.

$$X(\Omega) = e^{-\Omega^2/2\sigma^2} \quad (5.25)$$

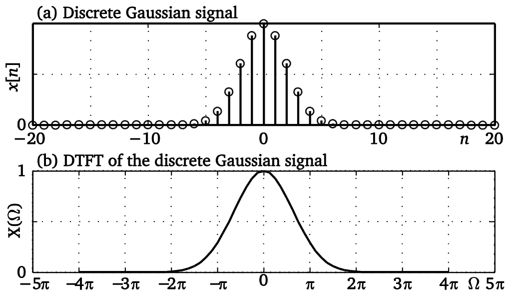


Figure 5.13 - DTFT of a Gaussian pulse for σ = 2. Note that all values are well contained within 3σ = 6.

Note that we would have to sample the Gaussian function by 4 times the maximum frequency in order to avoid significant aliasing. The reason is that this function has no obvious maximum frequency and no matter what we choose, the signal will still contain frequencies higher than that number albeit in low amplitudes. The alternate is to first filter the signal with an anti-aliasing filter before doing the DTFT.

Now the DTFT of periodic signals

All of the signals we looked at so far in this chapter were *aperiodic*, pulses standing alone. But what about discrete signals that are periodic? We have a transform for these as well and this is a yet one more type of Fourier transform. We call it the ***DTFT of periodic signal***. The DTFT of periodic signals is our most important type of Fourier Transform. Not because periodic signals are so important but because, the DTFT for periodic signals turns out to be *discrete*. This is our real goal. We want a discrete spectrum! We stated that as our goal. The DTFT of periodic signals, when modified slightly for finite length signals, gives the ***Discrete Fourier Transform (DFT)***, the most used form and for which the well-known Fast Fourier Transform algorithm was written. It took us a lot of pages in this book and 100's of years of history to get to this important point.

However, we are not quite there yet. Let's take a periodic but discrete-time signal with a period of N_0 samples and write its discrete Fourier series equation. Note we did not talk about a period when discussing DTFT of aperiodic signals, but we will now. Period now becomes relevant because these signals are periodic, so they have a period! And whenever, we have a period, the frequency resolution becomes discrete. However to derive a transform for periodic discrete signals, we have to go back to *discrete-time Fourier series* as out starting point.

The Fourier series is written in form of Fourier series coefficients for discrete-time signals as follows. (See chapter 3)

$$x[n] = \sum_{k=N_0} C_k \, e^{jk\Omega_0 n} \tag{5.25}$$

Where $\Omega_0 = 2\pi/N_0$ is the digital frequency of the discrete signal and N_0 is the period of the signal. The coefficients of the harmonics are given by

$$C_k = \frac{1}{N_0} \sum_{n=-N_0/2}^{N_0/2} x[n] \, e^{jk\Omega_0 n} \tag{5.26}$$

Since now we have N_0 samples of a periodic signal, we can indeed compute these coefficients. Let's take the DTFT of Eq. (5.26).

$$\begin{aligned} X(\Omega) &= \Im\left\{ \sum_{k=N_0} C_k e^{jk\Omega_0 n} \right\} \\ &= C_k \, \Im\underline{\underline{\left\{ \sum_{k=N_0} e^{jk\Omega_0 n} \right\}}} \end{aligned} \tag{5.27}$$

The coefficients are not a function of frequency, so they are pulled out in front. The DTFT of the underlined part, a summation of complex exponentials is a train of impulses.

$$\Im\left\{ \sum_{k=N_0} e^{jk\Omega_0 n} \right\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - k\Omega_0 - 2\pi m)$$

Substituting this expression into Eq. (5.27), we get the equation for the DTFT of a periodic signal.

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} C_k \, \delta\left(\Omega - \frac{2\pi k}{N_0}\right) \tag{5.28}$$

It may not be obvious here but the DTFT of a periodic discrete signal repeats the DTFS coefficients, C_k at every integer multiple of the digital frequency. That's what the second part, the impulse train is doing. This formulation is quite different from the DTFT of an aperiodic signal which we computed in Eq. (5.6) and repeat here.

$$X(\Omega)=\sum_{n=-\infty}^{\infty} x[n]\,e^{-j\Omega n} \qquad (5.29)$$

The DTFT in Eq. (5.28) for a ***periodic discrete-time signal*** tells us that the DTFT of a periodic signal consists of its ***DTFS coefficients repeated every*** N_0 ***samples***. Since N_0 is a finite number, the period of the signal, the samples are discrete and no longer continuous as they are for an aperiodic case. The spectrum is now *discrete*. Just what we like! Who wants to do integration when we have computers.

Repeating this important fact again: The *DTFT of both the aperiodic and the periodic signal repeats*. Looking at the spectrum, it appears as if the DTFT of a periodic signal is a sampled version of the DTFT of an aperiodic signal. Because of this, this case has come to be called the ***Discrete Fourier Transform***, known by the acronym DFT and often thought of as a sampled version of the DTFT.

In Fig. 5.14 we see the comparison of the CTFT and DTFT for a periodic signal. The CTFT of a continuous-time periodic signal is discrete but non-repeating. The DTFT for a discrete signal sampled at some frequency F_s, is discrete, however, it repeats at the sampling frequency F_s.

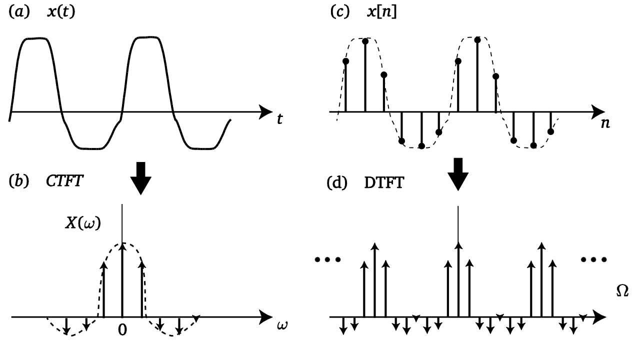


Figure 5.15 - Comparing CTFT and DTFT for periodic signals

Since the DTFT of a periodic signal is repeating DTFS coefficients, here we give a table of the DTFS coefficients for common signals. Knowledge of these makes computing the DTFT of periodic signals easy.

Table I
DTFS and DTFT of common function

	Time domain signal	**DTFS** $C_k = \frac{1}{N_0} \sum_{n=-N_0/2}^{N_0/2} x[n]\, e^{jk\Omega_0 n}$	**DTFT** $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$
1	$x[n] = 1$	1	$2\pi \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{N_0}\right)$
2	$x[n] = \delta[n]$ Impulse at 0	Does not exist	**1**
3	$x[n] = \delta[n - n_0]$ A shifted impulse	Does not exist	$e^{-j\Omega n_0}$
4	$x[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN_0]$ This is an impulse train of period N0.	$\frac{1}{N_0}$	$\frac{2\pi}{N_0} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{N_0}\right)$
5	$x[n] = e^{jn\Omega_0}$ A periodic complex exponential, with $\Omega_0 = \frac{2\pi}{N_0}$	$\begin{cases} 1 & k = mN_0 \\ 0 & elsewhere \end{cases}$	$2\pi \sum_{k=-\infty}^{\infty} \delta\ \Omega - \Omega_0 - 2\pi k$
6	Cosine, periodic	$\begin{cases} 1/2 & k = mN_0 \\ 0 & elsewhere \end{cases}$	$\pi \sum_{k=-\infty}^{\infty} \delta\ \Omega + \Omega_0 - 2\pi k$ $+ \pi \sum_{m=-\infty}^{\infty} \delta\ \Omega - \Omega_0 - 2\pi k$
7	Sine, periodic	$\begin{cases} 1/2 j & k = mN_0 \\ 0 & elsewhere \end{cases}$	$j\pi \sum_{k=-\infty}^{\infty} \delta\ \Omega + \Omega_0 - 2\pi k$ $- j\pi \sum_{k=-\infty}^{\infty} \delta\ \Omega - \Omega_0 - 2\pi k$

Now we look at a very basic discrete-time signal that is periodic. As we shall see, the DTFT instead of being continuous is discrete.

DTFT of a constant periodic function

$x[n] = 1; \quad -\infty < n < \infty$

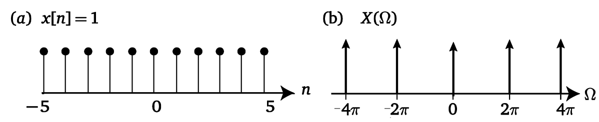


Figure 5.16 – DTFT of a constant discret signal

What we have for $x[n]=1$ is an impulse train of constant amplitudes given by sample number *n*. Hence we can write such a signal as

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-k]$$

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \qquad (5.32)$$

Recall from Chapter 4 that the amplitudes of the DTFT, is normalized value over one cycle and the real amplitude of the signal. The DTFT amplitudes just as the CTFT amplitudes are accurately related only to each other.

Specifying things in digital frequency is actually quite confusing. Instead of specifying the pulse train by the sample number *n*, we state that the time between each pulse is equal to T_0 , how does that change Eq. (5.32)? Each sample is T_0 seconds apart, which is the real-time period of this signal. We can convert the digital frequency into *real frequency* by dividing by this period as shown in Fig. 5.16. The units become radians per second.

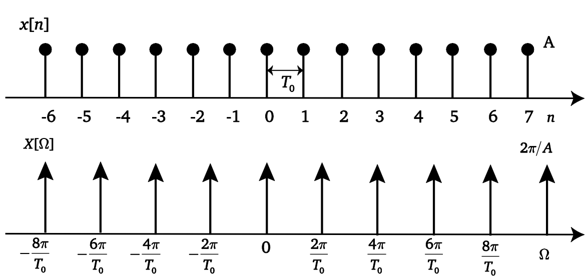


Figure 5.17 - DTFT of periodic pulse train

An alternate way to think about the frequency-domain impulse train is to see it as a set of infinite number of impulse pairs centered at the origin. The DTFT of each pair corresponds to a cosine. Hence we have an infinite number of cosines, each located at an integer multiple of the fundamental frequency, of ever increasing frequency.

DTFT of a sine wave

Here is an ultimate student-friendly function, discrete but periodic. Assume its frequency is ω_0.

$$x[n] = \sin(\omega_0 n)$$

The DTFT of this function is computed as follows. First we write the sinusoid as a sum of exponentials, then since we know that the DTFT of the sum of exponentials is an impulse train, we apply the result from Example 5.12.

$$x[n] = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$$

Now we write the DTFT as

$$X(\Omega) = \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\Omega - \omega_0 - 2\pi m) - \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\Omega + \omega_0 - 2\pi m)$$

Clearly these are impulse trains, with frequency ω_0 and then repeating with 2π . We draw them in Fig. 5.18. Each pair of impulses is located around integer multiple of ω_0 the frequency of the sinusoid. Then the whole thing repeats with 2π .

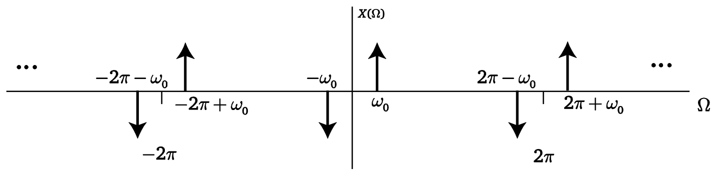


Figure 5.18 - DTFT of a sine wave

Summary of the Fourier transform types

In Fig. 5.19, we summarize four versions of the Fourier transform, for both periodic and aperiodic signals with both continuous and discrete time. Each has a unique behavior, however sharing many of the properties with each other.

Periodicity -> Time-resolution	Aperiodic	Periodic
Continuous-time (CT)	CTFT • Continuous Frequency resolution • Spectrum does not repeat.	CTFT • Discrete Frequency resolution • Spectrum does not repeat.
Discrete-time (DT)	DTFT • Continuous Frequency resolution • Repeating spectrum with 2pi.	DTFT • Discrete Frequency resolution • Repeating spectrum with sampling frequency.

Figure 5.19 – Four versions of the Fourier transform

Why is the Fourier series not mentioned in this figure? The reason is that Fourier transform can deal with all of these types of signals, *both periodic or non-periodic, as well as discrete or continuous-time signals*. These various versions of the Fourier transform can be used to accomplish the same thing as the Fourier series and more. The series are redundant to the periodic versions of the Fourier transform so in our education they have served their purpose as a starting point. We can put them away now and move on.

Table II - DTFT of common signals

Signal, $x[n]$	DTFT, $X[\Omega]$
$1, \quad -\infty < n < \infty$	$X[\Omega] = 2\pi \sum_{n=-\infty}^{\infty} \delta \;\; \Omega - 2\pi k$
$\operatorname{sgn}[n] = \begin{cases} -1 & 0 > n \\ 1 & 0 \leq n \end{cases}$	$\frac{1}{1-e^{-j\Omega}}$

$u[n]$	$\frac{1}{1-e^{-j\Omega}} + 2\pi \sum_{n=-\infty}^{\infty} \delta\ \Omega - 2\pi k$
$\delta[n]$	$1, \quad -\infty < \Omega < \infty$
$\delta[n-n_0], \ n_0 = \pm 1, \pm 2, \ldots$	$e^{-j\Omega n_0}$
$a\delta[n-n_1] + b\delta[n+n_2]$	$ae^{-j\Omega n_1} + be^{j\Omega n_2}$
$a^n\ u[n], \ \lvert a < 1 \rvert$	$\frac{1}{1-ae^{-j\Omega}}$
$e^{j\Omega_0 n}, \ \Omega_0 \ real$	$2\pi \sum_{n=-\infty}^{\infty} \delta\ \Omega - \Omega_0 - 2\pi k$
Square pulse of width τ , centered at n = 0	$\frac{\sin\left[\left(\frac{\tau}{2}\right)\Omega\right]}{\sin\ \Omega/2}$
Square pulse of width τ, centered at $n = n_0$	$\frac{\sin\left[\left(\frac{\tau}{2}\right)\Omega\right]}{\sin\ \Omega/2}\ e^{-j\Omega n_0}$

$\cos\ \Omega_0 n$	$\pi \sum_{n=-\infty}^{\infty} \left[\delta\ \Omega - \Omega_0 - 2\pi k\ + \delta\ \Omega + \Omega_0 - 2\pi k\ \right]$
$\cos\ \Omega_0 n + \phi$	$\sum_{n=-\infty}^{\infty} \left[e^{j\phi}\delta\ \Omega - \Omega_0 - 2\pi k\ + e^{-j\phi}\delta\ \Omega + \Omega_0 - 2\pi k\ \right]$
$\sin\ \Omega_0 n$	$\pi \sum_{n=-\infty}^{\infty} \left[\delta\ \Omega - \Omega_0 - 2\pi k\ - \delta\ \Omega + \Omega_0 - 2\pi k\ \right]$
$\sin\ \Omega_0 n + \phi$	$\sum_{n=-\infty}^{\infty} \left[e^{j\phi}\delta\ \Omega - \Omega_0 - 2\pi k\ - e^{-j\phi}\delta\ \Omega + \Omega_0 - 2\pi k\ \right]$

Summary of Chapter 5

1. The DTFT of a discrete-time aperiodic signal is developed by assuming that the period of the discrete pulse is infinitely long.
2. Because the period is presumed very long, the frequency resolution approaches zero, hence the DTFT, specified by $X(\Omega)$, becomes a continuous function of frequency.
3. The DTFT of an aperiodic signal is a function of the digital frequency Ω, which is unique only in 2π range. The DTFT computed around the 0 frequency is called the principal alias.
4. We need to compute the DTFT only in this range as DTFT in all other frequency ranges are identical to the principal alias.
5. The DTFT is given by

   $$X(\Omega)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

   The frequency Ω is continuous.
6. The iDTFT is given by

   $$x[n]=\frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

7. The DTFT of an aperiodic discrete signal is continuous and repeating.
8. The DTFT of a periodic discrete signal is given by

   $$X(\Omega)=2\pi\sum_{k=-\infty}^{\infty} C_k\ \delta\left(\Omega-\frac{2\pi k}{N_0}\right)$$

9. The DTFT of a periodic discrete signals is the discrete-time Fourier series coefficients, C_k repeating at the sampling frequency of the signal.
10. DTFT is most similar in behavior to discrete-time Fourier series, DTFS. The amplitudes however are normalized in DTFT.
11. The DTFT of a discrete periodic signal is discrete, with frequency resolution of $2\pi / N_0$ with N_0 equal to the samples per period.
12. DTFT leads us to the Discrete Fourier Transform (DFT) which can be used for finite length signals.