Intuitive Guide to Fourier Analysis

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Much of this book relies on math developed by important persons in the field over the last 200 years. When known or possible, the authors have given the credit due. We relied on many books and articles and consulted many articles on the internet and often many of these provided no name for credits. In this case, we are grateful to all who make the knowledge available free for all on the internet.

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1 | Trigonometric Representation of Continuous-time Periodic Signals

Jean-Baptiste Joseph Fourier
1768 – 1830

Jean-Baptiste Joseph Fourier was a French mathematician and physicist. He ws appointed to the École Normale Supérieure, and subsequently succeeded Joseph-Louis Lagrange at the École Polytechnique. He is best known for developing the Fourier series and its applications to problems of heat transfer and vibrations. The Fourier transform and Fourier's Law are also named in his honor. Fourier also did very important work in in the field of astronautics, as well as discovering the greenhouse effect for which he is not so well know. – From Wikipedia

What is Fourier analysis

When sunlight hits rain-soaked air, an interesting phenomena happens. Water drops take the ostensibly pure white light and split it into multiple colors. The mathematical description of this process, the subject of this book, was first tackled by Newton approximately 400 years ago. However, even though Newton was able to show that white light is in fact composed of other colors, he was unable to make the jump to the idea that light can be described as waves. He called the component colors of white light "specter" or ghosts, from which we get the word spectrum. It took the development of trigonometric series and the recognition that light can be thought of as a composite wave before Fourier could apply these ideas to the problem of heat transfer. Although the concept of harmonic trigonometric series already existed by the time he worked on the heat transfer problem, Fourier's contribution is considered so important that the whole field of trigonometric waveform analysis and synthesis now bears his name, *Fourier analysis*.

Fourier developed the following partial differential equation called the *Diffusion* equation to describe heat transfer through solids and other medium. Here v is a measure of heat and K, a heat diffusion constant of the material.

$$\frac{\partial v}{\partial t} = K \frac{\partial^2 v}{\partial x^2} \tag{1.1}$$

Fourier observed that the most general solution to this equation was given as a linear summation of sinusoids, i.e. sine and cosine waves of the form:

$$s(x) = \sum_{k=0}^{\infty} \left(a_k \sin kx + b_k \cos kx \right) \tag{1.2}$$

This equation led Fourier to conclude that an arbitrary wave can be represented as a sum of an infinite number of weighted sinusoids, i.e. sine and cosine waves. This trigonometric summation is now known as the **Fourier series**. This book is all about this simple but important idea.

Fourier analysis is applicable to a wide variety of disciplines and not just signal processing, where it is now an essential tool. Fourier analysis is also used in image processing, geothermal and seismic studies, stochastic biological processes, quantum mechanics, acoustics and even finance.

The Fourier analysis of waves or signals is similar to the concept of compound analysis in chemistry. Instead of atoms coming together to form a myriad of compounds, in signal processing sinusoids can be thought of as doing the same thing. A particular set of these sinusoids is called the fundamental set. Just as a compound may consist of two units of one element and four units of another, an arbitrary wave can consist of two units of one fundamental wave and four units of another. Hence we can create a particular wave by putting together the fundamental waves. This process is called the **Synthesis**. Conversely, the process of decomposing an arbitrary wave is termed the **Analysis**. These two complementary, linear processes fall under the name of **Fourier analysis** and its analog, the **Fourier transform**.

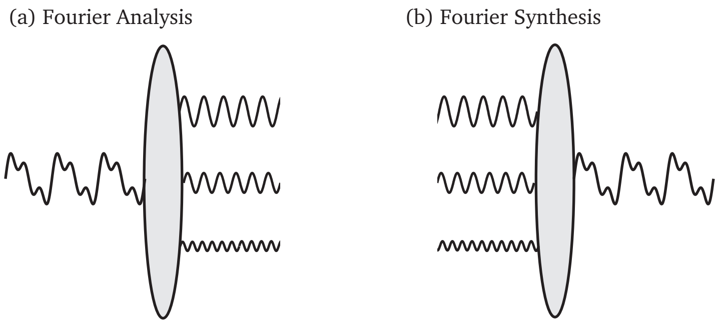

Figure 1.1: Fourier analysis is used to understand composite waves. (a) Analysis: breaking a given signal into sine and cosine components and (b) Synthesis: adding certain sine and cosines to create a desired signal.

Possibly it was the solution of Eq. (1.1) that led Fourier to notice that summation of harmonic sine and cosine waves leads to some interesting looking periodic waves. From this he posited that, conversely it is also possible to create any periodic signal by the summation of a particular set of harmonic sinusoids. This may not seem like a big thing now but it was a revolutionary insight at the time. Fourier's discovery was met with incredulity at first. Many of his contemporaries did not accept, and rightfully so, that his ideas did not apply to all signals. After some years of work by Fourier as well as other famous mathematicians of the age, his theorem was upheld, albeit not under all conditions and not for all types of signals. Subsequent development led to the Fourier transform, the extension of Fourier's original idea to non-periodic signals. But this computationally-demanding concept languished for over 100 years, until the development of the Fast Fourier Transform (FFT), by J.W. Cooley and John Tukey in 1965. The FFT, an algorithmic technique, made the computation of Fourier series simpler and quicker and finally allowed Fourier analysis to be recognized and used widely. It is now the premier tool of analysis in many fields.

Frequency and time domain views of a signal

Take the wave in Fig. 1.2. One would be hard pressed to guess its equation. Yet it is just a sum of three waves shown in Fig. 1.3(a), 1.3(b) and 1.3(c) of differing frequencies and amplitudes.

So although we know that the wave of Fig. 1.2 is created from the sum of three regular looking sinusoids of Fig. 1.3, how could we determine this, if we did not already know the answer?

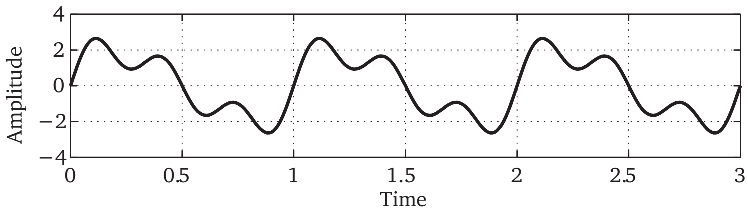

Figure 1.2: An arbitrary periodic wave. We would like to know its equation, but how? Fourier analysis does not give us the actual equation but allows us to recreate it with a fair amount of accuracy using simple sinusoids.

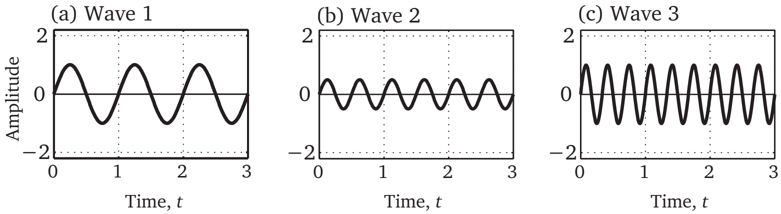

Figure 1.3: The components of the arbitrary wave of figure 1.2.

Spectrum of a signal

Let's look at Fig. 1.4 showing a wave in three dimensions. When we see a signal, we are looking at it in what is called, the **Time Domain**. What we are actually observing is a *component-summed* signal. Its components may be several single-frequency waves, which we are unable to see. In this three dimensional view, when we look at the signal from the side, each component appears apart on the frequency axis. We see the constituent frequencies (also called components) in this view. From this view, we see only a single vertical line at each of the discrete frequencies. This side view of the signal is called the **Frequency Domain**. Another name for this view is the **Signal Spectrum**.

The spectrum is a way to quantify the component frequencies. The spectrum of the signal of Fig. 1.2 is composed of just three frequencies and can be drawn as in Fig. 1.5(a). This is called a one-sided amplitude spectrum. The x-axis in Fig. 1.5(a) represents the frequencies of the signal and, the y-axis is the amplitude of those frequencies. In (a), we see the amplitude spectrum with amplitude of each frequency and in (b) we see the power spectrum, a more typical representation. For power spectrum, we plot instead of amplitude, the quantity amplitude-squared (which is equal to the instantaneous power), commonly shown in Decibels (dBs).

The graphical representation such as the one shown in Fig. 1.5 is a unique *signature* of the signal at a *particular time*. Most signals we work with are not such deterministic sums

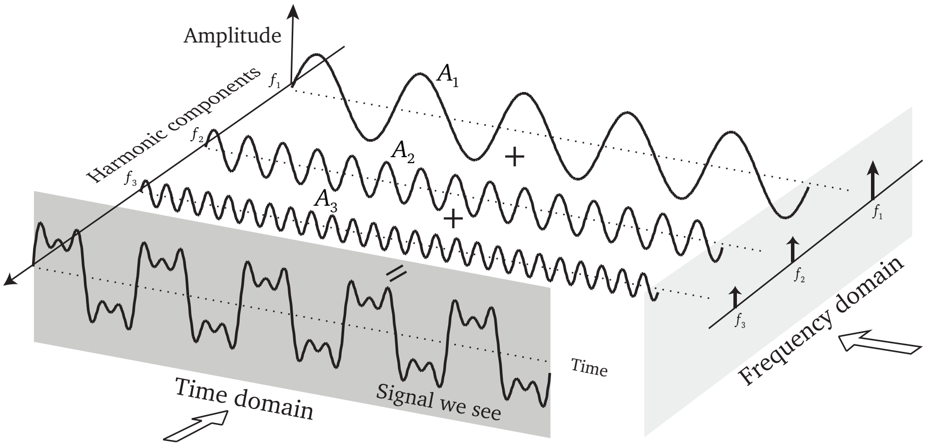

Figure 1.4: Looking at a composite wave from time and frequency perspectives gives a different view of the same signal.

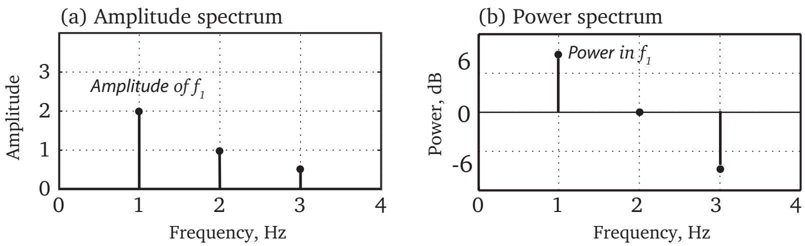

Figure 1.5: The frequency view of the arbitrary wave of Fig. 1.2. (a) Shows the amplitude and (b) shows the power in dBs.

of sinusoids, so a spectrum is a distinctive and variable quality of a signal. It is *not* a static thing but changes as the signal changes. Signals have distinctive spectrum and we can tell a lot about a signal by looking only at its spectrum. The signals for which Fourier analysis is considered valid have stable spectrum that do not change much over time. This property is generally called textbfStationarity.

Fundamental waves and their harmonics

The basic building blocks of Fourier analysis are a set of **harmonic sinusoids**, called the **basis set**. A basis set is our tinker-toy from which we can construct a variety of waves. The set contains an infinite number of sinusoids of differing frequencies related in a special way that we call *harmonic*.

We see a sinusoid of an arbitrary frequency in the first row of Fig. 1.6. Let's call this arbitrary frequency the *fundamental frequency*. We specify some waves based on this fundamental wave called the **Harmonics**. Each harmonic frequency is an integer multiple of the frequency of the fundamental. We see in this figure that the second wave has half the wavelength and twice the frequency of the first one and so on. Each k-th wave has a wavelength of T_0/k and a frequency of kf_0 with k being consecutive integers greater than 1. All such waves for $k > 1$ are called harmonics of the fundamental. The fundamental frequency is of course arbitrary, it can be anything, but its harmonics are strictly integer multiples of it.

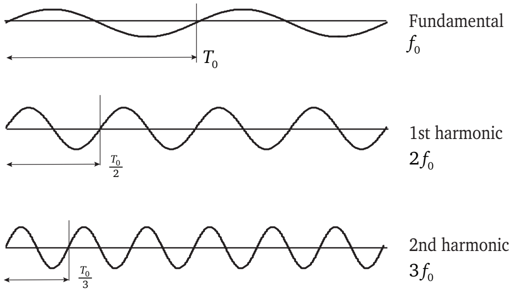

Figure 1.6: The fundamental and its harmonic.

Let's start with the expression of a complex sinusoid, a sine and a cosine of a certain frequency:

$$s(t) = \sin(2\pi f_0 t)$$
$$c(t) = \cos(2\pi f_0 t)$$

Here f_0 is an arbitrary frequency (measured in cycles/second or Hz). We will call this frequency the **fundamental**. The *basic period* T, of this sinusoid is inverse of the frequency. Note that if a sinusoid is periodic with period T, then it is also periodic with period $2T$, $3T$, etc. for integer multiples of the basic period. From this we get the definition of harmonic waves. A set of waves are harmonic if their frequency is an integer multiple of the fundamental wave's frequency. We can write this as a set.

$$\begin{aligned} s(t) &= \sin(2\pi f_k t) \\ c(t) &= \cos(2\pi f_k t) \end{aligned} \tag{1.3}$$

Here we have introduced an index k such that each harmonic frequency is equal to k times the fundamental frequency with k any arbitrary integer.

$$f_k = kf_0 \quad \text{for } k = 0, 1, 2, \ldots, \infty \tag{1.4}$$

In Fourier series formulation, the index k spans all positive integers to infinity, including zero. Let's rewrite the definition of the harmonics allowing the **phase** and the **amplitude** to vary. We give each harmonic a unique amplitude and phase, hence we rewrite the harmonic signals as:

$$\begin{aligned} s_k(t) &= A_k \sin(2\pi k f_0 t + \phi_k) \\ c_k(t) &= B_k \cos(2\pi k f_0 t + \phi_k) \end{aligned} \tag{1.5}$$

The wave $c_k(t)$ is a cosine wave of k-th harmonic frequency or kf_0, its amplitude being A_k and the phase in radians being ϕ_k. The signal $s_k(t)$ is a sine wave with similarly unique amplitude and phase for the same harmonic frequency, kf_0. Now although the frequencies are still related by multiples of k, we are allowing the amplitude and the phase of each harmonic to be different. Such waves are still harmonic.

What is amplitude? This is value of the wave's height at any particular time. For signals, it is often measured in volts. It can be positive or negative.

What is phase? A general sinusoid is defined by $x(t) = \text{A}\ sin(\omega_0 t + \phi)$. The term ϕ, a constant, is phase of the wave. We can think of it as the *place* the wave was at time, $t = 0$. A sine wave is defined by setting it as $\phi = 0$ radians. Cosine is defined by setting it to $\pi/2$ at t = 0. Think of phase as a type of delay or maybe even as a starting point. Once set, phase does not change for linear signals. Here we assume that phase and frequency are not a function of time.

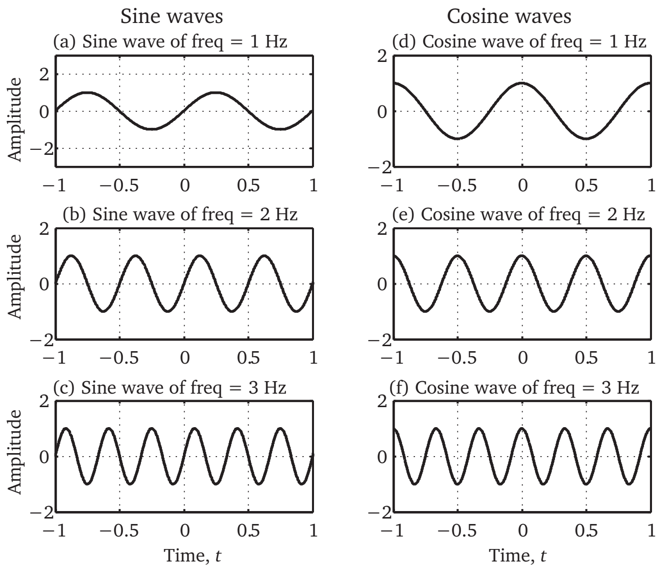

Figure 1.7: The fundamental of $f_0 = 1$ Hz and its two cosine and sine harmonics. All sines start at 0 and all cosines at peak amplitude.

In Fig. 1.7 we see two harmonics of a fundamental sinusoid of frequency of 1 Hz, with cosines on the right side and sine on the left side. All have peak amplitude of 1. Depending on your field of interest, the units of amplitude can be pretty much anything. In signal processing, the amplitude is measured in volts and phase in radians. The cosine always reaches its peak amplitude at time $t = 0$, and has a phase of $\pi/2$ at that time. The sine, no matter what the frequency, always has an amplitude of zero at time $t = 0$, which is equivalent to a phase of 0 radians. Hence no matter how many sine waves we add together such as in Eq. (1.5) they will never achieve any amplitude other than zero at time = 0, hence can represent only those waves which are zero-valued at time = 0.

Harmonics as basis functions

In Fig. 1.8 we add together three sine and three cosines with the phase of 0 radians and amplitude of 1. When we add these three waves, we see that cosines in (a) add such that the peak is equal to 3. This is an *even* wave. The cosines hence add *constructively* at time = 0 and at other times that are integer-period seconds away. The sum of three sine waves added together in (b) however, looks strange. This asymmetry is a consequence of the sine wave being an *odd* wave. In (c), we add the three sine and the three cosine together. The behavior of this summation defies an easy explanation.

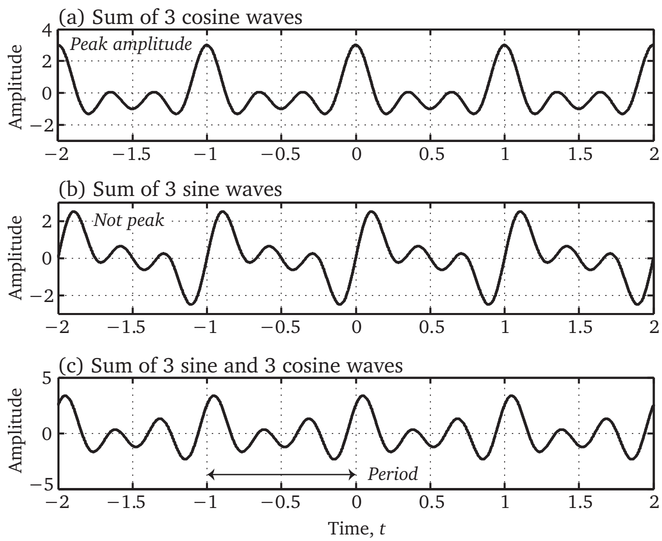

Figure 1.8: Sums of constant amplitude harmonics. (a) Sum of all cosine start at peak summed amplitude of 3. (b) Sum of sines starts at 0 (c) and hence summation of all starts at 3 also.

To examine this effect further, we add together an even larger number, 20 cosine and 20 sine harmonics, each of the same amplitude and 0 phase. In Fig. 1.9(a) we can clearly see that the cosine sum is symmetric about time = 0 point. Similarly in (b), we see that the sine are non-symmetric. Now we add the magnitude (the peak-to-peak amplitude) of the waves per the following equation.

$$s(t) = \sum_{k=0}^{\infty} \left| \cos(k\omega_0 t) + \sin(k\omega_0 t) \right|$$

In this equation, we define the radial frequency as $\omega_0 = 2\pi k f_0$. The summation of harmonic sine and cosine of equal amplitudes gives a function that is approaching an impulse train, i.e a signal consisting of very narrow pulses. This result in Fig. 1.9(c) gives us a clue to an important property of sinusoids and we will make use of it in subsequent chapters. This property says that a summation of an infinite number of harmonic sinusoids (sine and cosine) results in an impulse train.

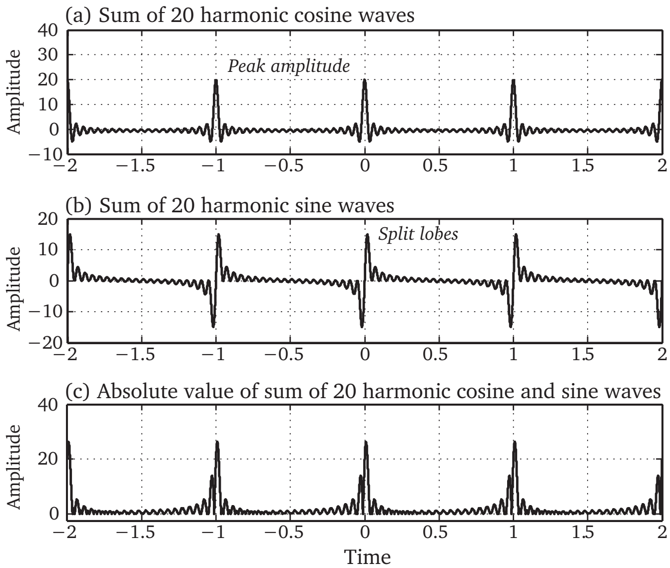

Figure 1.9: Sums of large numbers of both sine and cosine harmonics approach an impulse train. Note that the sum of cosines in (a) creates an even blip, where for sines it is odd as in (b).

What if we allow the amplitudes and phases of each sinusoid to vary? An example of a wave created using this idea is shown in Fig. 1.10 where each sine and cosine has a different amplitude and phase. Note that the frequency of the composite wave is equal to the frequency of the fundamental, which is 1 Hz.

$$s(t) = 0.1\cos(2\pi t - 0.5) + 0.3\cos(4\pi t) - 0.4\cos(6\pi t - 0.1) \\ - 0.5\sin(2\pi t + 0.1) - 0.8\sin(4\pi t - 0.3) + 0.67\sin(6\pi t + .19) \quad (1.6)$$

The most important thing to note is that by adding any number of harmonics, and allowing the amplitudes and phase of each to vary, we can create or mimic many other waves. In Fig. 1.10, we have an example of just one such "interesting" looking wave created by using only three different sinusoids of distinctly different amplitudes and phases. This is exactly the idea behind Fourier synthesis and analysis.

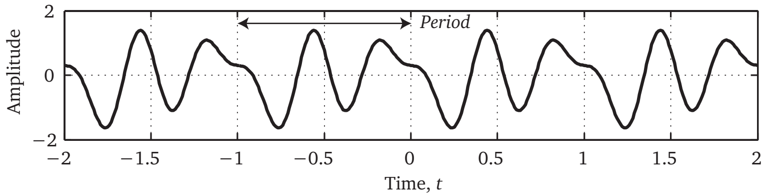

Figure 1.10: A wave composed of arbitrary amplitude harmonics of Eq. (1.6) *begins to look "interesting".*

Even and oddness of sinusoids

Sinusoids have a lot of interesting properties. There are a few that are important in Fourier analysis. One important property of harmonic sine and cosine waves is the symmetry or the *oddness* of the wave. All sine waves are considered odd functions because they obey the following definition of an odd function.

$$f(x) = -f(-x) \tag{1.7}$$

If you look at a sine wave, you see that it starts with an amplitude of 0 at time = 0. If we were to flip it about the y-axis, the images would not overlap. But if one of the sides was first flipped about the x-axis, then they do overlap. That is a description of *oddness* of signals. It requires two flips for values to coincide, as we can see from the two negative signs in Eq. (1.7).

The cosine waves on the other hand are called *even* functions by a similar definition. The two sides of a cosine wave, if flipped about the y-axis, would overlap, hence there is only one negative in the equation below for even symmetry.

$$f(x) = f(-x) \tag{1.8}$$

By the superposition principle, if multiple odd waves are summed together, the resulting wave will remain odd. If multiple even waves are summed, then the resulting wave will remain even. A mixture will have no symmetry. This becomes important when synthesizing, which is the process of putting some waves together to make a desired wave. If a wave to be synthesized is purely odd, or an even function, then it will only contain sine, or cosine, depending on its symmetry.

Making waves

Square waves

Now we try to create some common waveforms using this idea of harmonic summation. In Fig. 1.11, we attempt to create a square wave by adding together harmonic sinusoids. Since square wave of Fig. 1.11(a) is an odd function we will need only sine waves to create it. Fig. 1.12(a) has the same wave shifted so that it is even. This one can be created using only cosine. For both cases, we start with a wave of the same frequency as the square wave, as the square wave's fundamental frequency. This seems like a good start. But why? Because a periodic wave created from the addition of harmonics will always be periodic with the period of the fundamental. We then add more harmonic sine waves for the odd version of the square wave in Fig. 1.11(a) and cosine waves for the even version in Fig. 1.12(a) and watch the evolution of the square wave. With only three terms in the addition result in a pretty decent looking square wave. Each addition of a sine wave (or a cosine) with a specific frequency

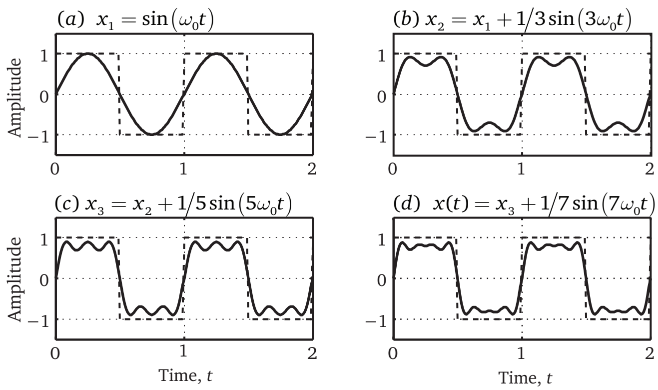

Figure 1.11: Synthesizing an odd square wave by adding odd harmonics of differing amplitudes. (a) start with fundamental of 1 Hz, (b) add to it harmonic of frequency 3 Hz, (c) add harmonic of frequency 5 Hz, in (d) add another harmonic of 7 Hz which looks quite good compared to underlying square wave. The "quantity", which is the amplitude of each harmonic added keeps getting smaller.

and amplitude makes the synthesized wave appear closer to a square wave. We are in-fact

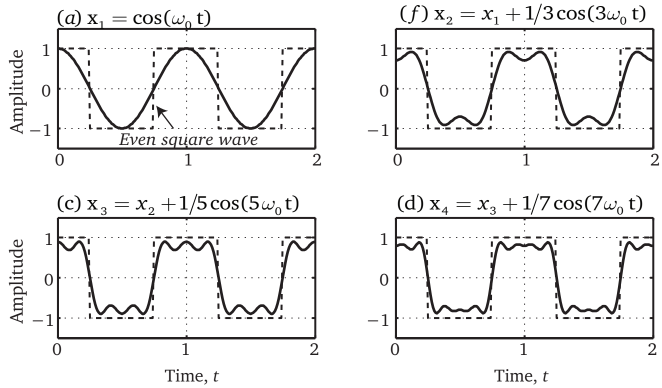

Figure 1.12: Synthesizing an odd square wave by adding even harmonics of differing amplitudes.

a math-chef who is creating recipes for making all kinds of interesting waves using specific "quantities" of sinusoids. And by quantity we mean the varied amplitudes and the phases of each harmonic. Collectively, the amplitude and the phase of a particular harmonic is called its **coefficient**.

Here are the recipes for the two types of square wave, an even and an odd. The ingredient list is limited, as we used only three terms beyond the fundamental.

$$x_1 = \sin(\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) + \frac{1}{5}\sin(5\omega_0 t) + \frac{1}{7}\sin(7\omega_0 t)$$
$$x_2 = \cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \frac{1}{5}\cos(5\omega_0 t) - \frac{1}{7}\cos(7\omega_0 t)$$

We note that in both of these "recipes" only odd harmonics are used to create this square wave. This is true for both the odd and even versions of the square wave. This is an interesting thing to know. The reason why we use only odd harmonics is that sinusoids of even harmonics cancel the odd harmonics as we see in Fig. 1.13 and hence mixing of odd and even harmonics does not allow us to create useful waves. We see in Fig. 1.13 that the two even harmonics of frequency 2 and 4 Hz are poles apart from the odd harmonics of frequency 1 and 3 Hz (at t = .5 sec.). We find that summation of consecutive harmonics begins to approach an impulse-like signal, as we see in Fig. (1.9) due to this destructive quality.

The Square wave is probably the single most important signal in signal processing and we will keep coming back to it in each chapter. The summation with k as the odd integer can go from $+\infty$ to $-\infty$ to form a really great looking square wave. The more terms we add, the closer we get to what we are trying to achieve. But in fact for square waves, we are never able to create a perfect square wave, as Gibbs phenomena takes hold near the corners. The corners never do become the true right angles as we would wish. This tells us

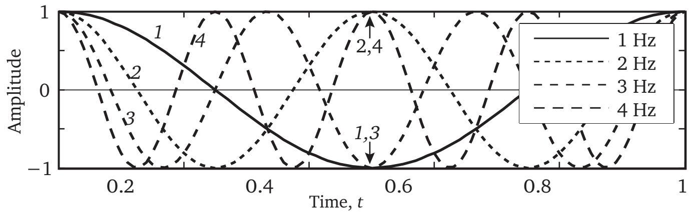

Figure 1.13: Even order harmonics are destructive and hence are not used in combination with odds to create most communications signals such as square waves.

that this form of harmonic representation, despite our best efforts, may not result in a perfect reconstruction for every signal. A square wave is one of those signals.

Gibbs phenomenon

In Fig. 1.14, we see that even as k, the number of terms in the square wave summation is increased, the oscillation at the corner points never goes away. This behavior called the **Gibbs phenomenon** is a clear demonstration of the fact that Fourier representation can not mimic all periodic waves. Hence waveforms that have hard discontinuities in amplitude, are avoided in signal processing. Instead of true square waves, we use shaped waves with gentle corners and curves which are easier to represent with sinusoids.

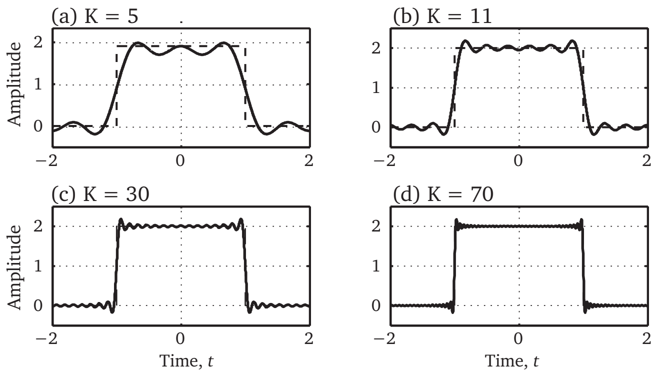

Figure 1.14: Gibbs phenomena does not allow for a perfect Fourier representation of a square wave.

Creating a sawtooth wave

Let's look at one more special signal, a sawtooth wave. The sawtooth wave is an odd function hence composed only of sine waves. We give its equation as

$$\text{sawtooth}(t) = \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin(2\pi k f_0 t)}{k} \tag{1.9}$$

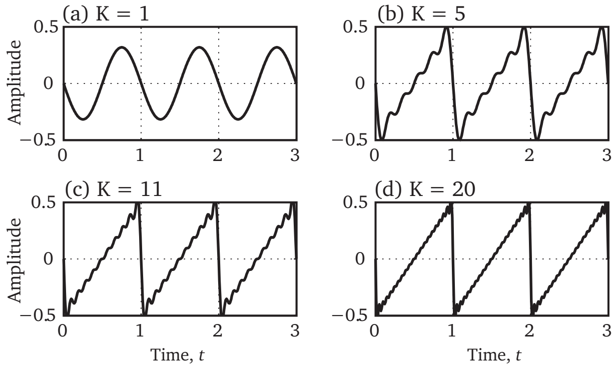

Figure 1.15: Evolution of a sawtooth wave

Note that in Eq. (1.9) the coefficient is inversely proportional to the harmonic index and hence the contribution of higher frequencies is decreasing. We write this equation allowing the index to get as large as possible. In real life, we can often get by with just a few terms.

Generalizing the Fourier series equation

A Fourier series is a general equation consisting of the summation of weighted harmonics, where by manipulating the weighting (hence coefficients) we can represent many different periodic waves. We called it a recipe but its mathematical name is **Fourier representation**.

$$f(t) = \sum_{k=1}^{K} a_k \cos(2\pi f_k t) + \sum_{k=1}^{K} b_k \sin(2\pi f_k t) \tag{1.10}$$

The coefficients a_k represent the coefficient of the k-th cosine wave and b_k of the k-th sine wave. What is K? This is the largest harmonic index we use in any particular summation. If we use only 10 terms (or harmonics), then $K = 10$. We can control this parameter depending on the accuracy desired. For a general representation, we set K to ∞.

The sum of sine and cosines are always symmetrical about the x-axis so there is no possibility of representing a wave with a DC offset using the form in Eq. (1.10). (*The term DC comes from direct current but it is used in signal processing to mean a constant.*) To create a wave of a non-zero mean, we must add a new term to Eq. (1.10). The constant, a_0 is added in Eq. (1.11) so we can create waves that can move up (or down) from the x-axis.

$$\boxed{f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(2\pi f_k t) + \sum_{k=1}^{\infty} b_k \sin(2\pi f_k t)} \tag{1.11}$$

Eq. (1.11) is called the **Fourier series equation**. The coefficients a_0, a_k, b_k are called the **Fourier Series Coefficients (FSC)** . The process of Fourier analysis consists of computing these three types of coefficients, given an arbitrary periodic wave, $f(t)$.

Multiple ways of writing the Fourier series equation

We find several different forms of the Fourier series equation in literature, and that makes understanding this equation confusing. One common representation is by using the radial frequency $\omega_k = 2\pi f_k$ to make the equation simpler to type. We can write this form of Fourier series as

$$\boxed{f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(\omega_k t) + \sum_{k=1}^{\infty} b_k \sin(\omega_k t)} \tag{1.12}$$

Now we define T_0 the period of the fundamental frequency.

$$T_0 = \frac{1}{f_0}$$

Then the period of the k-th harmonic becomes T_0/k and its frequency, $f_k = k/T_0$. We can alternately write the Fourier series equation by adopting this form of the frequency in Eq. (1.13).

$$\boxed{f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi k}{T_0} t\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi k}{T_0} t\right)} \tag{1.13}$$

This is an another way to write-out the Fourier series representation. We specifically added the DC term, now we look at a way to get rid of it. We can shorten the Fourier series equation by starting at zero frequency, hence index k starts at 0 instead of at 1. Now the DC term disappears as it is included as the zero frequency coefficient obtained by setting the index $k = 0$. The DC term is now included as the 0-th coefficient. Now we have this new form with the DC term gone.

$$\boxed{f(t) = \sum_{k=0}^{\infty} a_k \cos(2\pi f_k t) + b_k \sin(2\pi f_k t)} \tag{1.14}$$

We can simplify even more. We can get rid of the sine terms in Eq. (1.12) altogether. We know that sine and cosine are really the same thing, one is just the shifted version of the other. The representation in Eq. (1.12) can be written solely with cosine by shifting each sine wave by π. This way sine becomes a cosine and we only need cosine in the Fourier series, making it more concise.

$$\boxed{f(t) = a_0 + \sum_{k=1}^{\infty} c_k \cos(\omega_k t + \phi_k)} \tag{1.15}$$

Each harmonic, whether a sine or a cosine can be thought of as really a cosine of some phase. The sine is actually a cosine with shifted phase (or start point) and can be written as $\sin(k\omega_0 t) = \cos(k\omega_0 t + \phi)$. In expanded form Eq. (1.15) will look like this. Now not only are the amplitudes variable but so are the phases and the whole expression uses only the cosine waves.

$$f(t) = c_0 + c_1 \cos(\omega_1 t \pm \phi_1) + c_2 \cos(\omega_2 t \pm \phi_2) + c_3 \cos(\omega_3 t \pm \phi_3) + \ldots$$

Fourier series in the complex form

In its *most* important representation, the *complex representation*, the Fourier series is written as

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi \frac{k}{T_0} t}} \tag{1.16}$$

Here we introduce a new term, called the **complex exponential**. It even looks complex, $e^{j2\pi \frac{k}{T_0} t}$. It can represent both a sine and a cosine by change of the sign of the exponent. In the next chapter we will discuss this function in detail. The expanded form of the Fourier series in terms of the complex exponential looks like this.

$$f(t) = C_0 + C_1 e^{j2\pi \frac{1}{T_0} t} + C_2 e^{j2\pi \frac{2}{T_0} t} + C_3 e^{j2\pi \frac{3}{T_0} t} + \ldots$$

This form says that we can create a periodic function by summing together complex exponentials. Although the complex form of the Fourier series is scary looking, it is the most used form. In next chapter, we will look at how it is derived and why we use it in Fourier analysis.

All these different representations of the Fourier series are **identical** and mean exactly the same thing. They are all the many ways you see Fourier representation in books.

The Fourier Analysis

The process of adding together a bunch of sinusoids to create useful waves is called the Synthesis process. Synthesis of waveforms is of course interesting but what is really useful is to take an arbitrary periodic signal and figure out its components. Sort of like trying to figure out what the ingredients of a particular dish. By ingredients, we mean frequencies in the signal that contain significant power or amplitudes. This is the main use of Fourier analysis. It is called, not surprisingly, the **Analysis** part. What this involves is to make a guess of the fundamental frequency f_0, and then computing the amplitudes (coefficients) of a certain number of harmonics. It is no guarantee that the fundamental we choose will result in finding all the main signal components exactly. But in most cases, we have a pretty good idea of signal components a-priori. So the process works well enough.

The usefulness of the process can be seen in the equation of the sawtooth wave in Eq. (1.9). Fourier series allows us to create an estimate of the wave using a few or a lot of terms. Hence Fourier series represents an *estimate* of the true representation, which can have any number of terms.

The Fourier analysis process consists of finding the **series coefficients**. When we talk about Fourier series coefficients (FSC), we are talking about the amplitudes of the sine and cosine harmonics, and nothing else. Once we decide on a fundamental frequency, a starting point for the analysis, we already know all the harmonic frequencies since they are integer multiples of the fundamental frequency. All we have to do now is to compute these Fourier coefficients.

1. Coefficients of the cosine $a_k \cos(2\pi k f_0 t)$ with $k = 1, 2, 3, \ldots, \infty$.
2. Coefficients of the sine $b_k \sin(2\pi k f_0 t)$ with $k = 1, 2, 3, \ldots, \infty$.
3. The DC offset or the coefficient of the 0-th frequency, $k = 0$.

We will look at each of these three types of coefficients separately and see how to compute them.

Computing a_0, the DC coefficient

We are given an arbitrary periodic signal, $f(t)$ of period T. Fourier series says that the signal $f(t)$ is composed of a summation of K sinusoidal harmonics. Our task is to find the coefficients of each of these harmonics starting with $k = 0$ to $k = K$.

$$f(t) = \boxed{a_0} + \sum_{k=1}^{\infty} a_k \cos(2\pi f_k t) + b_k \sin(2\pi f_k t) \tag{1.17}$$

The constant a_0 in the Fourier series equation represents the DC offset. If our target wave has a nonzero DC component (if its average amplitude value is not zero), then we know that $a_0 \neq 0$. But before we compute it, let's take a look at another useful property of sine and cosine waves. Both sine and cosine waves are symmetrical about the x-axis. When you integrate a sine or a cosine wave over one period, you always get zero. The area above the

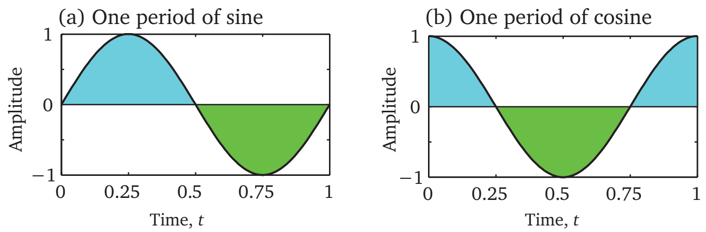

Figure 1.16: The area under both sine and cosine over one period is zero, no matter what their frequency.

x-axis cancels out the area below it. This is always true over one period as we can see in Fig. 1.16. The same is also true of the sum of sine and cosine. Any wave made by summing sine and cosine waves also has zero area over one period. If we were to integrate the given signal $f(t)$ over one period as in Eq. (1.17), the area obtained will have to come from the coefficient a_0 only. None of the sinusoids make any contribution to the integral and they will all fall out. Hence the calculation of the DC term becomes easy.

$$\int_0^{T_0} f(t)dt = \int_0^{T_0} a_0 dt + \cancelto{0}{\int_0^{T_0} \left(\sum_{k=1}^{\infty} a_k \cos(2\pi f_k t) + b_k \sin(2\pi f_k t) \right) dt}$$

Now we integrate the signal, $f(t)$ over the fundamental period T_0.(We can start anywhere, since the signal is periodic.) The second term is zero in Eq. (1.18), since it is just the integral of a wave made up of a sum of sine and cosine which individually integrate to zero over one period. We compute a_0 by computing the integral of the wave over one period. The area

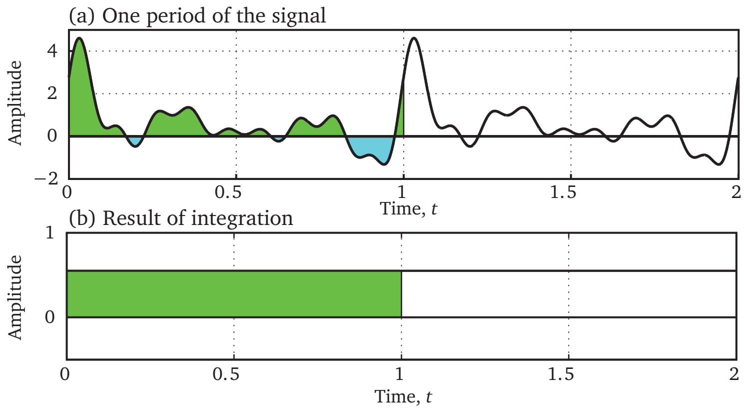

Figure 1.17: (a) The periodic signal before integration. (b) After integration of exactly one period, only the DC component is left.

under one period of this wave is equal to

$$\int_0^{T_0} f(t)dt = \int_0^{T_0} a_0 dt$$

Integrating this simple equation, we get,

$$\int_0^{T_0} f(t)dt = a_0 T_0$$

We can now write a very easy equation for a_0

$$a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)dt \tag{1.18}$$

Summary: To compute the DC coefficient, we integrate $f(t)$ over one period. (What is that period? It is the period of the fundamental.) The result of the integration is equal to the 0-th coefficient.

Computing b_k, the coefficients of sine waves

Here we employ a different trick from trigonometry to compute the coefficients ($k > 0$) of the sine waves. Below we show a sine wave that has been multiplied by itself.

$$f(t) = \sin(n\omega_0 t) \times \sin(n\omega_0 t)$$

We notice that the product of the two waves lies entirely above the x-axis and has a net

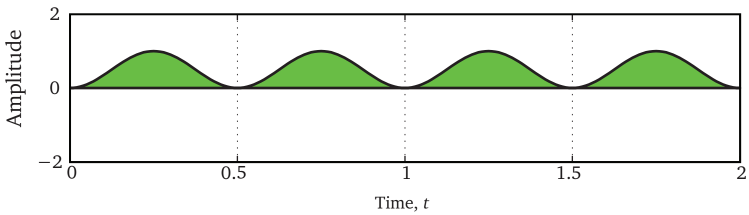

Figure 1.18: A sine wave multiplied by itself has non-zero area under one period.

positive area. From integral tables we can compute this area as equal to

$$\int_0^{T_0} b_k \sin(k\omega_0 t) \times \sin(n\omega_0 t)dt = b_k \frac{T_0}{2}$$

Now multiply the sine wave by an arbitrary harmonic of itself to see what happens to the area.

$$f(t) = \sin(\omega_0 t) \times \sin(\underline{m}\omega_0 t)$$

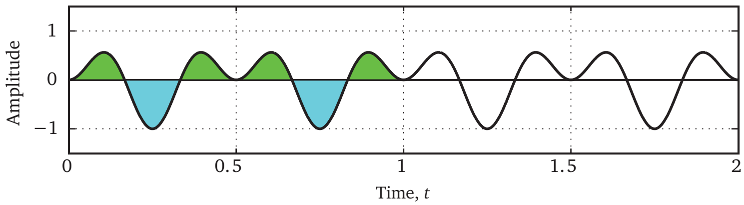

Figure 1.19: A sine wave multiplied by its harmonic has zero area under one period.

We get an important observation that the area in one period of a sine wave multiplied by *any* of its harmonics is zero. We conclude that when we multiply a signal by any of its harmonics, and integrate the product over one period, then the contribution belongs to just that harmonic and none others. All other harmonics contribute nothing. Writing this in integral form,

$$\int_0^{T_0} b_k \sin(k\omega_0 t) \times \sin(m\omega_0 t) dt = 0 \quad \text{for } k \neq m.$$
$$\int_0^{T_0} b_k \sin(k\omega_0 t) \times \sin(m\omega_0 t) dt = \frac{T_0}{2} b_k \quad \text{for } k = m.$$

The same is true of cosine.

$$\int_0^{T_0} a_k \cos(k\omega_0 t) \times \cos(m\omega_0 t) dt = 0 \quad \text{for } k \neq m.$$
$$\int_0^{T_0} a_k \cos(k\omega_0 t) \times \cos(m\omega_0 t) dt = \frac{T_0}{2} a_k \quad \text{for } k = m.$$

We see that the result of the integration of the product of two harmonics when their frequencies are unequal is **zero**. It is non-zero only when the waves have the *same* frequency. Hence if we multiply our signal by its harmonics, one at a time, the result we get for each case is the coefficient of the harmonic being used for multiplication with the signal. We can do that k times, multiplying the signal by the k-th harmonic, then integrating to get the amplitude of that harmonic hidden in the signal.

Let's multiply a sine wave by a cosine wave to see what happens. We get an another important result; the area under the product when multiplying a sine and a cosine is equal to zero whether the frequencies are the same or not. This is also the concept of orthogonality. We say that these waves are orthogonal as they contribute nothing to the integral.

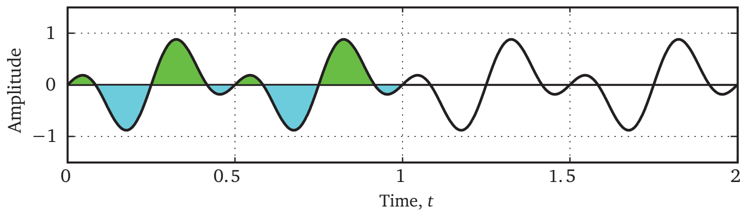

Figure 1.20: A sine wave multiplied by a cosine has total area of zero under one period.

Summarizing

$$
\begin{aligned}
&\int_0^{T_0} b_k \sin(k\omega_0 t) \times \sin(m\omega_0 t)dt = 0 \quad \text{for } n \neq m \\
&\int_0^{T_0} b_k \sin(k\omega_0 t) \times \sin(m\omega_0 t)dt = \frac{T_0}{2} b_k \quad \text{for } n = m \\
&\int_0^{T} b_k \cos(n\omega_0 t) \times \sin(m\omega_0 t)dt = 0
\end{aligned}
\tag{1.19}
$$

Another very satisfying interpretation of these properties is that sine and cosine waves act as filtering signals. In essence they act as a narrow-band filter and ignore all frequencies except the one of interest. This is the fundamental concept of a filter.

Here are the key results.

1. If you multiply a sine or a cosine wave by any of its harmonics, the area under the product is zero.
2. If you multiply a sine or a cosine of a particular frequency by itself, the area under the product is proportional to the Fourier coefficient of that frequency.
3. The area under a sine wave multiplied by a harmonic cosine is always zero. (Because sine and cosine are orthogonal!)

We use this information to compute the b_k coefficients. Successively multiply the target signal, $f(t)$ by a sine wave of a specific harmonic and then integrate over one period as in equation below.

$$
\begin{aligned}
\int_0^{T_0} f(t)\sin(k\omega_0 t)dt = \cancelto{0}{\int_0^{T_0} a_0 \sin(k\omega_0 t)dt} + \int_0^{T_0} b_k \sin(k\omega_0 t)\sin(k\omega_0 t)dt \\
+ \cancelto{0}{\int_0^{T_0} a_k \cos(k\omega_0 t)\sin(k\omega_0 t)dt}
\end{aligned}
\tag{1.20}
$$

We know that the integral of the first and the third term in Eq. (1.20) is zero since the first term is just the integral of a sine wave multiplied by a constant and the third term is of a sine wave multiplied by a cosine wave. This simplifies our equation considerably. We know that the integral of the second term is

$$\int_0^{T_0} b_k \sin(k\omega_0 t)\sin(k\omega_0 t)dt = \frac{b_k T_0}{2}$$

From this we obtain b_k as follows

$$\boxed{b_k = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(k\omega_0 t)dt} \tag{1.21}$$

The coefficient b_k is hence computed by taking the target signal over one period, successively multiplying it with a sine wave of k-th harmonic frequency and then integrating. This gives the coefficient for that particular harmonic. If we do this K times, we get K individual b_k coefficients.

Computing coefficient of cosine, a_k

The process of computing the coefficients of the cosine harmonics is the same as one used for the sine waves. Instead of multiplying the target signal, $f(t)$ by a sine wave we multiply the target signal by a cosine wave of frequency, $k\omega_0$.

$$\begin{aligned}\int_0^{T_0} f(t)\cos(k\omega_0 t)dt = \cancelto{0}{\int_0^{T_0} a_0\cos(k\omega_0 t)dt} + \cancelto{0}{\int_0^{T_0} a_k\sin(k\omega_0 t)\cos(k\omega_0 t)dt} \\ + \int_0^{T_0} a_k\cos(k\omega_0 t)\cos(n\omega_0 t)dt\end{aligned}$$

Now terms 1 and 2 become zero. The third term is equal to

$$\int_0^{T_0} a_k\cos(k\omega_0 t)\times\cos(n\omega_0 t)dt = \frac{a_k T_0}{2}$$

and the coefficient can be written as

$$\boxed{a_k = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(k\omega_0 t)dt} \tag{1.22}$$

So the process of finding the coefficients is multiplying the target signal with successively larger harmonic frequencies of a cosine wave and integrating the results. This is easy to do in software. The result obtained is the coefficient for that specific frequency of sine wave. We do the same thing for cosine coefficients. Conceptually the process of computing the coefficients consists of filtering the target signal, one frequency at a time. In software when we compute the spectrum, this is exactly what we are doing hence a spectrum can be seen as tiny little filters that pull out the quantity of that harmonic in the signal. And by quantity we mean the amplitude of the harmonic.

Example 1.1. Find the Fourier series coefficients for the function:

$$f(t) = \big(\cos(2\pi t)\big)^3$$

From Fig. 1.21 we can immediately spot two pieces of information. The first is that the period the wave is 1 second long (its frequency is 1 Hz), and second, that the symmetry of the signal is even. Because it is an even signal, we know that it only contains cosine components. (But of course that makes sense as the function is cosine cubed.)

First we calculate the a_0 coefficient to determine the DC offset.

$$a_0 = \int_0^1 \big(\cos(2\pi t)\big)^3 dt = 0$$

This result agrees with the graph. There is symmetry about the x-axis meaning there is no DC offset. Now using Eq. (1.22) we can determine the a_k coefficients (of cosine) starting with the fundamental harmonic. The fundamental frequency f_0 is 1 Hz.

$$a_1 = 2\int_0^1 \big(\cos(2\pi t)\big)^3 \cos(1 \times 2\pi t) dt = 0.75$$

Here is the plot of only the a_1 coefficient cosine wave. It is clear that one coefficient is not

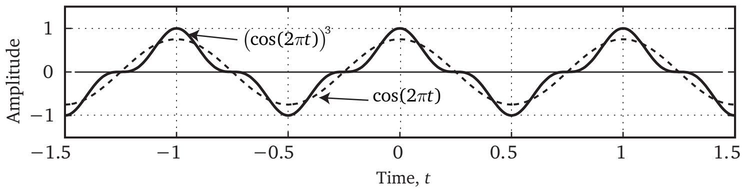

Figure 1.21: $\big(\cos(2\pi t)\big)^3$ solid curve and its first order representation.

enough to fully represent the signal. We calculate a few more coefficients.

$$a_2 = 2\int_0^1 \left(\cos(2\pi t)\right)^3 \cos(2\times 2\pi t)dt = 0$$
$$a_3 = 2\int_0^1 \left(\cos(2\pi t)\right)^3 \cos(3\times 2\pi t)dt = 0.25$$

With $a_0 = 0$, $a_1 = .75$, $a_2 = 0$, and $a_3 = 0.25$ the series is given as

$$f(x) = .75cos(2\pi t) + .25cos(6\pi t)$$

Using trigonometric identities this can be proven to be equal to $\cos^3(2\pi t)$. If we continue to calculate more a_k coefficients (for $k > 3$) we will see that they are all zero after a_3. From this we can say that function cosine cubed is in fact made up of only two cosines of frequency 1 and 3 Hz. In this case, Fourier series representation is an exact representation of the signal, but this is not always the case.

Example 1.2. Find the Fourier series coefficients for the function:

$$f(t) = \sin^2(2\pi t) + \sin(2\pi t)$$

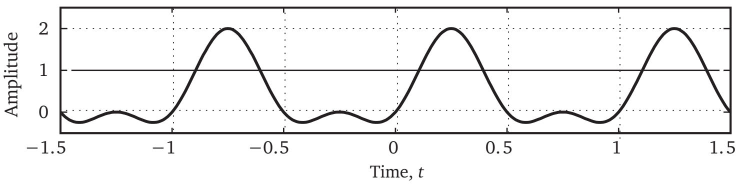

Figure 1.22: $\sin^2(2\pi t) + \sin(2\pi t)$.

Using trigonometric identities we can rewrite the function as:

$$f(t) = \frac{1}{2} + \sin(2\pi t) - \frac{1}{2}\cos(4\pi t)$$

From this we can expect to have a non-zero a_0 coefficient for the DC offset, a a_2 cosine coefficient and a b_1 sine coefficient. The values of these will match the coefficients of the signals' equation. The results of the integrals show that this is indeed the case. The Fourier representation in this case is exact. This only happens if the original signal it composed only of harmonic sinusoids. In real life, this situation is unlikely and hence our Fourier representation is usually only an approximation.

$$a_0 = \int_0^1 \big(\sin^2(2\pi t) + \sin(2\pi t)\big)dt = \frac{1}{2}$$
$$a_1 = 2\int_0^1 \big(\,\sin^2(2\pi t) + \sin(2\pi t))\cos(1\cdot 2\pi t)\big)dt = 0$$
$$a_2 = 2\int_0^1 \big(\,\sin^2(2\pi t) + \sin(2\pi t))\cos(2\cdot 2\pi t)\big)dt = -\frac{1}{2}$$
$$b_1 = 2\int_0^1 \big(\,\sin^2(2\pi t) + \sin(2\pi t))\sin(1\cdot 2\pi t)\big)dt = 1$$
$$b_2 = 2\int_0^1 \big(\,\sin^2(2\pi t) + \sin(2\pi t))\sin(2\cdot 2\pi t)\big)dt = 0$$

Coefficients are the spectrum

How do we go from coefficients to a spectrum? Assume that we have a signal which we have analyzed and have found that it has nine harmonics with $f_0 = 2.5$ Hz. The coefficients of the nine harmonics starting with $k = 0$ are given by

$$b_k = [.4\ -.3\ .7\ .7\ .3\ .275\ .25\ .2\ .2]$$
$$a_k = [.25\ .2\ .4\ .5\ -.2\ .2\ .1\ -.05\ .02]$$

here b_k are coefficients of the sine harmonics and a_k are the coefficients of the cosine harmonics. We plot these in a bar graph in Fig. 1.23. Both sine and cosine of same frequency are plotted next to each other for the same harmonic. This is a spectrum that in essence displays the recipe for the signal. It tells us how much of each harmonic i.e. its amplitude we need to recreate the signal.

The Fourier series coefficients tell us *how much* of each harmonic frequency is contained in our target signal, so they are a measure of its recipe. The coefficients for this reason are analogous to the *spectrum* of the signal. Commonly a spectrum is in terms of "Power" but the coefficients we compute via Fourier analysis are "not" power. They are *amplitudes*. The term spectrum is often used synonymous with power spectrum. One needs to know the type of spectrum one is plotting or looking at.

In signal processing, the coefficients computed for cosine are called *Real* and those for sine, *Imaginary*. Of course, there is nothing imaginary about the coefficients of sine, they are just as real as the cosine coefficients. It is just one of the many confusing terms we

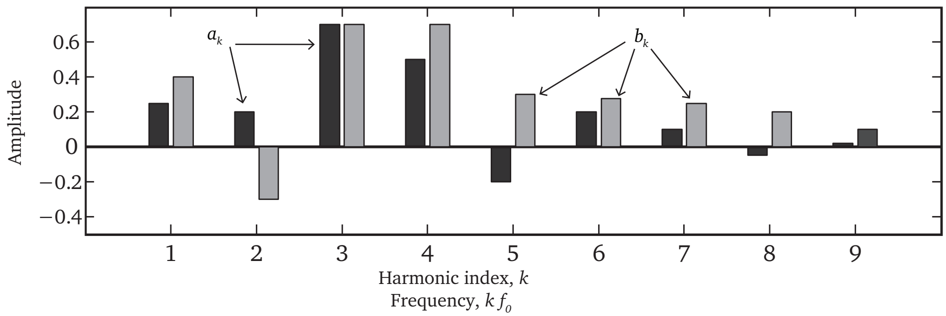

Figure 1.23: The coefficients of the harmonics.
For each harmonic index, k representing a frequency of kf_0, there are two coefficients, one for sine and the other for cosine. These coefficients can be positive or negative and represent a formula for creating a Fourier representation of the test wave.

come across in signal processing. You may now rightly say that the spectrum in Fig. 1.23 in terms of sine and cosine coefficients, is not the way we see a spectrum in books. A spectrum ought to have just one number for each frequency. The coefficients by themselves provide frequency content information but are hard to understand. Instead, we often *combine* the sine and cosine coefficients into two more real world terms, called the magnitude and the phase.

Since harmonic sine and cosine are orthogonal to each other, and have the same frequency, we can compute their *sum* by thinking of them as vectors. We can compute this by doing the root-sum-square (RSS) of their coefficients. This RSS amplitude term is called the ***Magnitude***. Now we plot the modified spectrum using the magnitude on the y-axis. The new coefficient, the magnitude is a one number for each frequency and is given by

$$\boxed{c_k = \sqrt{a_k^2 + b_k^2}} \tag{1.23}$$

While the amplitude can indeed be negative, the magnitude is always positive. The effect of the sign of the amplitude is now seen in a term called phase, which we calculate by

$$\boxed{\phi_k = \tan^{-1} \frac{b_k}{a_k}} \tag{1.24}$$

We plot these on two separate plots, one for the magnitude and second for the phase calculated from Eq. (1.24).

Magnitude spectrum is by far the more preferred form. The phase spectrum usually does not offer useful information, and often goes ignored. Confusingly, most people make

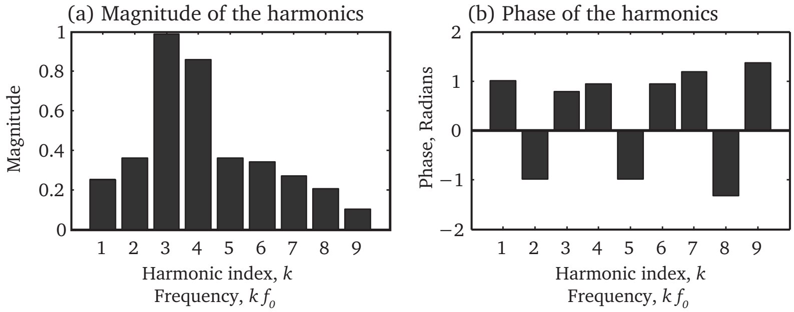

Figure 1.24: The Magnitude and Phase spectrum.
In (a), we see the RSS of both the sine and cosine coefficients, representing the magnitude of that harmonic frequency. In (b), we compute the phase for each harmonic. The magnitude on the right is usually quite instructive but phase is hard to comprehend as it changes quickly from one frequency to the next.

no distinction when talking about amplitude or magnitude, often these two terms are used interchangeably. The first form, the spectrum based on the real and imaginary coefficients, changes depending on the starting point of the analysis, where the second form, the magnitude and the phase are not a function of the starting point. Hence practicing engineers prefer the second from.

The process of doing the Fourier analysis consists of computing the amplitude of each harmonic and then from it the magnitude and the phase. In Fig. 1.25 we show a power spectrum with y-axis given in dBs and shows the power of each harmonic, not the magnitude. To convert a magnitude spectrum to a power spectrum, we use Parseval's theorem. The tells us that the total power in a signal is the sum of the powers in each harmonic. The power in each harmonics is defined as the square of its magnitude. Hence to represent power, you square each magnitude value and then compute its dB value by $10 \cdot log(c_k)^2$. Alternately, you can just compute the log10 of the coefficient and multiply it by 20.

$$\text{Power in the } k\text{-th harmonic} = 20\log(c_k) \tag{1.25}$$

The power spectrum is often normalized to maximum power, such as bin 3 here. (The bin is a form of identifier of the harmonic.) The level of each component is the dB equivalent of its ratio to the maximum power. All component levels relative to the maximum power, are said to be a certain number of dBs below the maximum. The maximum level is zero dB, with all other values shown as negative. These values are also called **Power Spectral density** (PSD) because they are a form of density of the power across a bandwidth.

Looking at the Fourier series coefficients, note that there are just K (the largest k) number of such coefficients, one for each of the harmonics. The spectrum from Fourier series

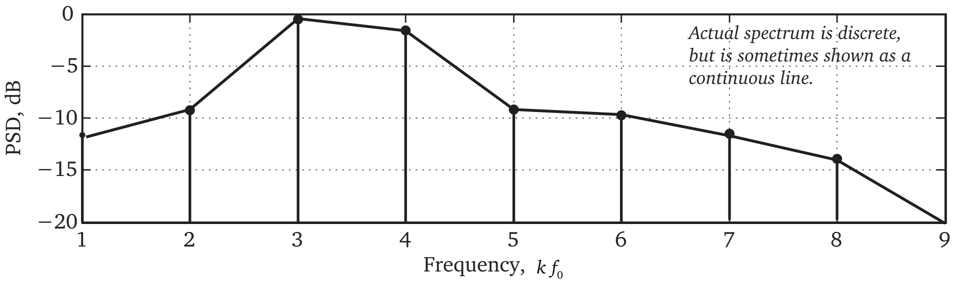

Figure 1.25: A traditional Power spectrum created from the Fourier coefficients.

coefficients (FSC) is hence considered *discrete*. In time domain the signal is continuous but in frequency domain, the one-sided spectrum developed from the FSC is discrete. There are only K terms. This is an important and significant property of the Fourier series, a one-sided spectrum consisting only of a finite number of components, each a unique value. The x-axis can be drawn either by just giving the index k, but more commonly the actual frequency ranging from 0, f_0, $2f_0$, $3f_0$,...,etc..

Fourier analysis applies only to periodic waves. But for real signals we can never tell what the period is nor where it starts. No periodicity can usually be seen. In fact, a real signal may not be periodic at all. In this case, further developments of the theory allow us to extend the "period" to infinity so we just pick any section of a signal or even the whole signal and call it "The Period", representing the whole signal. This idea is the basis of Fourier transform which we will discuss in Chapter 4.

In this chapter we talked about the Fourier series representation of an arbitrary periodic wave in terms of weighted harmonic sine and cosine waves. In the next chapter, we look at how we can do the same using the confusing and scary-looking complex exponentials.

Summary of Chapter 1

In this chapter, we introduced the concept of using sinusoids to represent an arbitrary periodic wave. We also introduced the concept of the fundamental frequency and its harmonics. The Fourier series representation consists of finding unique weightings of these harmonics to represent a particular periodic wave. These weightings are called the Fourier series Coefficients (FSC). These coefficients when plotted as a function of the frequency represent the spectrum of the signal. The spectrum calculated using FSC is discrete although the signal is continuous in time.

Terms used in this chapter:

- **Fundamental frequency** - the smallest frequency of the signal to be represented by Fourier series.
- **Harmonic frequencies** - All integer multiples of the fundamental frequency.
- **Sinusoids** - sine or cosine wave.
- **Harmonic coefficients** - The amplitude of a harmonic.
- **Real and imaginary** - Cosine is said to exist in the real plane and sine in the imaginary plane.
- **Magnitude** - the RSS value of the amplitudes of the sine and cosine of a particular harmonic. It is always positive.
- **Phase** - the stating value of a wave at t = 0, often specified in radians.

1. The most common trigonometric form of the Fourier series is given by

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(2\pi f_k t) + \sum_{k=1}^{\infty} b_k \sin(2\pi f_k t)$$

2. The coefficients of the Fourier series are easily computed by

$$a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)dt$$
$$b_k = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(k\omega_0 t)dt$$
$$a_k = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(k\omega_0 t)dt$$

3. Many periodic signals can be represented by weighted sum of harmonics sinusoids. The representation is an estimation and may not be an exact replication.
4. Harmonic sinusoids are orthogonal to each other, hence the integral of their products (or cross product) is zero.

5. The linearity property of the Fourier series implies that a change of the coefficients of one harmonic does not effect the coefficients of the other harmonics.
6. A time or phase shift of the signal does not effect the magnitude of the coefficients.
7. To synthesize a signal based on Fourier series, we pick a fundamental frequency first. All harmonics are integer multiples of the fundamental.
8. We designate harmonics by letter k. Hence for all integer k, all kf_0 frequencies are harmonics of the fundamental frequency f_0.
9. The Fourier analysis means to find the Fourier coefficients of the Fourier series.
10. The Fourier coefficients are discrete since the harmonics are discrete. Hence the spectrum of a periodic continuous-time signal is discrete.
11. A spectrum can be a spectrum of amplitude, magnitude or of power. These are not the same thing.

Questions

1. Can you state in words the principal behind Fourier series.
2. What is the third harmonic of a sinusoid of frequency 3 Hz.
3. Can any signal have harmonics?
4. Why is one harmonic orthogonal to another. What trigonometric property tells us that this is true?
5. If we add three non-harmonic sinusoids together, is the resulting signal periodic?
6. If we multiply the amplitude of a signal by a non-linear value, what effect will that have on its frequency?
7. A change in phase of a cosine wave means it is still orthogonal to a sine wave of the same frequency, true or false?
8. What is the maximum amplitude of N harmonic cosine waves added together. What is it for sine waves.
9. We need to represent a wave that starts at time $t = 0$. What type of harmonics will be in its representation.
10. Summation of odd and even waves can be used to create any waveform we want, true or not?
11. Is Fourier series representation an accurate representation of a wave? Why not?
12. Can we create a Fourier series representation of any wave?
13. Why do we consider the set of harmonics a *basis set*? What constitutes a basis set?
14. Sine and cosine waves are a basis set for Fourier analysis. Can you give an example of another set of basis functions.
15. What quality of sine and cosine makes them suitable as a basis set?
16. Fourier series analysis is considered a linear process. Why?

17. What do the coefficients of a Fourier series represent? What does the a_0 coefficient represent?
18. What is the Fourier series representation of this signal? $s = A + B\sin(2\pi f t)$.
19. We want to compute the FSC coefficients of this signal. $s = sin(6.5t) - cos(4.75t)$. What should we pick as the fundamental frequency?
20. If the target wave is shifted by a certain phase, what happens to its coefficients?
21. How many coefficients would you need to describe this wave? $x(t) = 2 + B\sin(4\pi t + \pi/2) - cos(12\pi t)$. Find the coefficients of this above signal.
22. What is the fundamental period of this FS representation. $x(t) = 2/\pi(sin(4\pi t) + (1/3)sin(12\pi t) + (1/5)sin(20\pi t) + (1/7)sin(28\pi t)$. What are the coefficients of the first four cosine and sine coefficients.
23. Given these equations, what are the Fourier series coefficients, a_0, a_1, a_2 for each case.
    (a) $y = \frac{1}{2} + \frac{3}{4}\sin(\pi x) - \frac{3}{5}\cos(2\pi x)$.
    (b) $y = \frac{3}{4}\cos(2\pi x) - \frac{3}{5}\cos(3\pi x)$.
24. What is the difference between amplitude and magnitude?
25. The amplitude of a harmonics varies from -1 v to +1 v. What is the magnitude of the harmonic? What is the power of the harmonic? What is the value of the power in dBs?
26. Examine Fig. 1.11 and give first four coefficients of the even square wave.