Intuitive Guide to Fourier Analysis

Charan Langton
Victor Levin

Much of this book relies on math developed by important persons in the field over the last 200 years. When known or possible, the authors have given the credit due. We relied on many books and articles and consulted many articles on the internet and often many of these provided no name for credits. In this case, we are grateful to all who make the knowledge available free for all on the internet.

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2 | Complex representation of continuous-time periodic signals

Leonhard Euler
1707 - 1783

Leonhard Euler was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function. Euler is considered to be one of the greatest mathematicians to have ever lived. A student of Johann Bernoulli, Euler was the foremost scientist of his day. Born in Switzerland, he spent his later years at the University of St. Petersburg in Russia. He perfected plane and solid geometry, created the first comprehensive approach to complex numbers. Euler was the first to introduce the concept of $\log x$ *and* e^x *as functions and it was his efforts that made the use of* $e, \imath$ *and* π *the common language of mathematics. Among his other contributions were the consistent use of the trigonometric sine, and cosine functions and the use of a symbol for summation. A father of 13 children, he was a prolific man in all aspects, languages, medicine, botany, geography and physical sciences and has left his mark on our scientific thinking.– From Wikipedia*

Euler's equation

$$\boxed{e^{j\omega t} = \cos\omega t + j\sin\omega t}$$

This is the famous Euler's equation. Bertrand Russell and Richard Feynman both gave this equation plentiful praise with words such as "the most beautiful, profound and subtle expression in mathematics" and "the most amazing equation in all of mathematics." This perplexing equation was first developed by Euler (pronounced Oiler) in the early 1800's.

The $e^{j\omega t}$ in Euler's equation is a decidedly confusing concept. What exactly is the role of j in $e^{j\omega t}$? We know from algebra that it stands for $\sqrt{-1}$ but what is it doing here with the sine and cosine? Can we even visualize this function?

The complex exponential

The function $e^{j\omega t}$ goes by the name of **complex exponential** (CE). This function is of the greatest importance in signal processing and Fourier analysis. We are going to discuss its conceptual nature and its application to Fourier analysis.

$$e^{j\omega t} = \underline{\cos\omega t + j\sin\omega t} \tag{2.1}$$

In Eq. (2.1), the complex exponential is on the left side and its sinusoid equivalent expression on the right. Ignore for now the complex exponential $e^{j\omega t}$ on the left hand side and examine the right hand side of this equation, containing the sine and cosine waves.

We can plot this function by assigning a value to ω, and then for a range of t, calculating both $\cos\omega t$ and $\sin\omega t$ values. Since the value of ω is a constant, we have three values now, t the independent time variable and associated sin and cos values calculated at t. With these three values, we can create the 3-D plot shown in Fig. 2.1. Time is plotted on the x-axis, and the values of the two sinusoids on the other two axes, creating a three-dimensional figure of a helix.

The expression for the negative exponential (with negative in the exponent) is written as

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t \tag{2.2}$$

The difference between $e^{j\omega t}$ and $e^{-j\omega t}$ can be seen in the Fig. 2.1(a) and (b) in that the two helix seem to be rotating in the opposite directions. The negative exponentials are said to rotate in the counter-clockwise direction and the positive exponentials rotate in the clockwise direction. We can plot this graph in Matlab using this code.

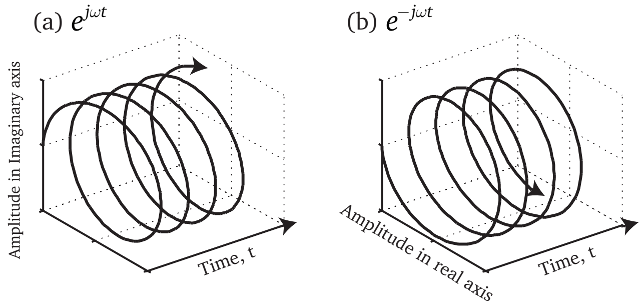

Figure 2.1: $e^{j\omega t}$ when plotted looks like is a helix. It is a 3-D function of three values, time t, the independent variable, and then for a fixed frequency ω, the values, $sin(\omega t)$ and $cos(\omega t)$ on the other two axes. The exponent of the CE indicates direction of advance or movement, (a) positive exponent and (b) negative exponent.

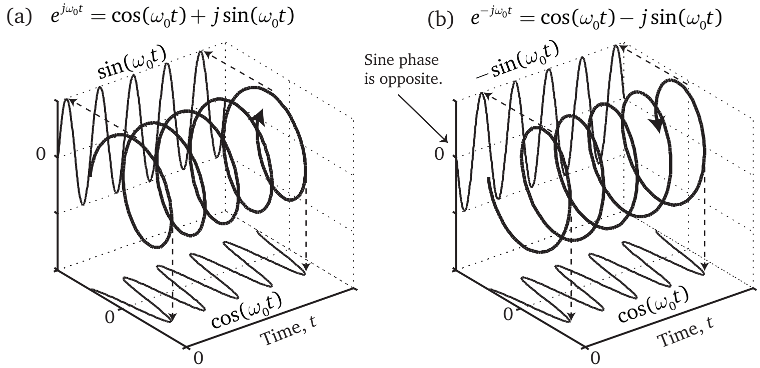

Figure 2.2: The projections of the complex exponential are sinusoids. (a) $e^{j\omega t}$ and its two projections, (b) $e^{-j\omega t}$ and its two projections. Note that the sine wave in this projection has different phase than one for the positive CE in (a). That is the only difference between the two CEs.

Projections of the complex exponenetial

Since CE is a complex function, we examine its projections on the real and the imaginary axes. In Fig. 2.2(a) we plot the projections of the helix on the *Real* and the *Imaginary* planes. In Cartesian terms, these would be called the (X, Y) and (Z, Y) planes. The projections of the complex exponential on these two planes are sine and cosine waves. The Real projection of the complex exponential is a cosine wave and Imaginary projection is a sine wave.

For the negative exponent, or the so called negative complex exponential, the sine wave is flipped 180° degrees from the positive exponential as we see in Fig. 2.2(a). Often this exponential is referred to as having a negative frequency, however it is not really the frequency that is negative. From the definition of the negative exponent exponential in Eq. (2.2) we see that negative sign of the exponential results in the imaginary projection, the sine wave doing a 180° phase change, or equivalently being multiplied by -1.

The *Real part* of the negative as well as the positive complex exponential is a positive cosine wave.

$$Re\left(e^{-j\omega t}\right) = \cos\omega t \qquad Im\left(e^{-j\omega t}\right) = -\sin\omega t$$

The imaginary part of the positive exponential is a positive sine but is negative for the negative CE.

$$Re\left(e^{j\omega t}\right) = \cos\omega t \qquad Im\left(e^{j\omega t}\right) = \sin\omega t$$

The negative exponential has as its imaginary part a negative sine wave. The positive exponential has a positive sine as its imaginary part. The real part, which is a cosine, is same for both. We don't see any negative frequencies here, an idea generally associated with the negative complex exponential.

Now that we have these two forms of the exponentials, let's do some math with them. Adding and subtracting the complex exponentials, $e^{j\omega t}$ and $e^{-j\omega t}$, and then after a little rearrangement, we get these new ways of expressing a sine and a cosine.

$$\begin{aligned}\frac{1}{2}\left(e^{+j\omega t} + e^{-j\omega t}\right) &= \frac{1}{2j}\left(\cos\omega t + \cancel{j\sin\omega t} - \cos\omega t + \cancel{j\sin\omega t}\right)\\ &= cos(\omega t)\end{aligned} \tag{2.3}$$

$$\begin{aligned}\frac{1}{2j}\left(e^{+j\omega t} - e^{-j\omega t}\right) &= \frac{1}{2j}\left(\cancel{\cos\omega t} + j\sin\omega t - \cancel{\cos\omega t} + j\sin\omega t\right)\\ &= sin(\omega t)\end{aligned} \tag{2.4}$$

Let's see graphically what Eq. (2.3) and Eq. (2.4) look like. When we plot the two composite exponentials, we get the two plots in Fig. 2.3. The first figure shows that this composite exponential has a real projection of a cosine and the second, only the sine. The helix is gone, it has collapsed into a cosine and a sine. Hence the sine and cosine can be said to be composed of these two complex exponentials.

We are so used to thinking of sine and cosine as sort of atomic functions. It seems hard to believe that they can be created by adding other functions. But Eq. (2.3) and Eq. (2.4) tell us that both sine and cosine can be created by adding complex exponentials. How can that be, when the CEs are 3-D functions? This is a case of two 3-D functions coming together

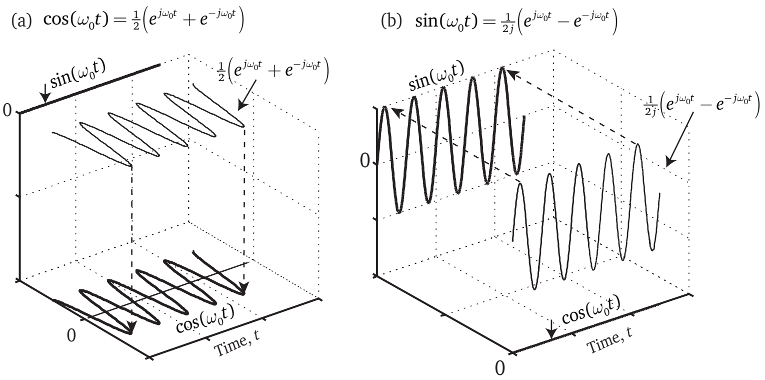

Figure 2.3: (a) Plotting $(e^{+j\omega t} + e^{-j\omega t})/2$ gives a cosine wave with zero projection on the Imaginary plane (b) Plotting $(e^{+j\omega t} - e^{-j\omega t})/2j$ gives us a sine wave with zero projection on the real axis.

to create a 2-D sinusoid. This sounds strange but it's actually not an unfamiliar concept. We can add two 2-D functions and get a 1-D function. An example is when we add a sine and a 180° shifted sine, we get a straight line, a 1-D function. So a 2-D function created by two 3-D functions should not be a big stumbling block.

The sinusoids

So how did Euler's equation come about and why is it so important to signal processing. We will try to answer that by first looking at Taylor series representations of the exponential e^x, sines and the cosines. The Taylor series expansion for the two sinusoids is given in 2-D by the infinite series as

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \end{aligned} \tag{2.5}$$

Note that each one of these series is composed of many individual exponential functions. So sine waves really are composed of exponentials! However these are ***real*** exponentials that are non-periodic and not the same thing as the complex exponentials. Complex exponentials are a special class of real exponentials and are used as alternate to the sinusoids in Fourier analysis, since they are periodic and offer ease of expression and calculation which is not obvious at first.

Real exponentials are used in Laplace analysis as the basis set instead of the complex exponentials we use in Fourier analysis. Real exponentials are far more general than sinusoids and complex exponentials and allow analysis of non-periodic and transient signals,

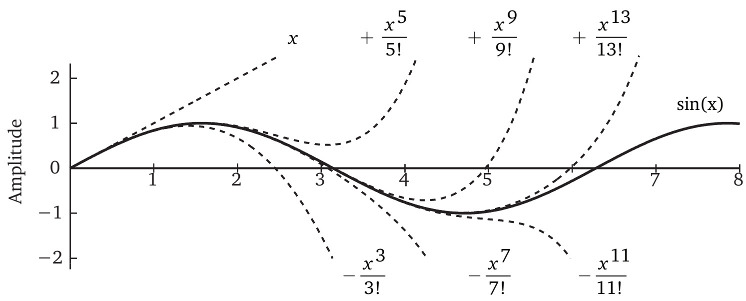

Figure 2.4: Sine wave as a sum of many exponentials of different weights.

something we are not going to cover in this book. Laplace analysis is a general case of which Fourier is a special case applicable only to special types of periodic or *mostly* periodic signals.

Taylor series expansion for the exponential e^x gives this series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \tag{2.6}$$

All three of these equations Eq. (2.5) and Eq. (2.6) are straight forward concepts. And indeed if we plot these functions, we would get just what we are expecting, the exponential of e and the sinusoids. How close our plots come to the continuous function depends on the number of terms that are included in the summation.

The similarity between the exponential and the sinusoids series in Eq. (2.5) and (2.6) shows clearly that there is a relationship here. Now let's change the exponent in (2.6) from x to $j\theta$. Note we will use the term θ here instead of ωt to keep the equation concise. Now we have by simple substitution, the expression for $e^{j\theta}$ as

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \dots \tag{2.7}$$

We know that $j^2 = -1$ and $j^4 = 1$, $j^6 = -1$, etc., substituting these values, we rewrite this series as

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{j\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{j\theta^7}{7!} + \dots \tag{2.8}$$

We can separate out every other term with j as a coefficient to create a two-part series, one without the j and the other with

$$\begin{aligned} e^{j\theta} &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \qquad \text{This is cosine} \\ &+ j\theta - \frac{j\theta^3}{3!} + \frac{j\theta^5}{5!} - \frac{j\theta^7}{7!} + \dots \qquad \text{This is } j \text{ times sine} \end{aligned} \tag{2.9}$$

We see that first part of the series is a cosine per Eq. (2.5) and the second part with j as its coefficient is the series for a sine wave. Hence we showed that

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

We can now derive some interesting results like the following. By setting $\theta = \pi/2$, we can show that

$$\begin{aligned} e^{j\pi/2} &= \cancel{\cos(\pi/2)} + j\sin(\pi/2) \\ &= 0 + j \cdot 1 \\ &= j \end{aligned}$$

By setting $\theta = 3\pi/2$, we can show that

$$\begin{aligned} e^{j3\pi/2} &= \cancel{\cos(3\pi/2)} + j\sin(3\pi/2) \\ &= 0 + j \cdot -1 \\ &= -j \end{aligned}$$

And an another interesting result

$$\begin{aligned} e^{j\pi} &= \cos(\pi) + j\sin(\pi) = -1 \\ \Rightarrow e^{j\pi} + 1 &= 0 \end{aligned}$$

The purpose of this exercise is to convince you that indeed the complex exponential is special sum of only a sine and a cosine. The function still retains a wondrous and mysterious quality, with added tinge of fear. However, we need to get over our fear of this equation and learn to love it. The question now is why bring up the Euler's equation in context of Fourier analysis? Why all this rigmarole about the complex exponential, why aren't sines and cosines good enough?

In Fourier analysis, we computed the coefficients of sines and cosines (the harmonics) separately. We also discussed the three different formulations of the Fourier series using sines and cosines, then only with cosines and then with complex exponentials. Fourier analysis using the trigonometric form is not easy in practice. Trig functions are easy to understand but hard to manipulate. Adding and multiplying them is a pain. On the other hand, doing math with exponentials is considerably easier. (See examples in Appendix A.)

Using a single exponential can simplify math in Fourier series. This is the main advantage of switching to complex exponentials in using the complex form of the Fourier series.

The math looks hard but is actually easier. However complex exponentials bring with them some conceptual difficulties. They are hard to visualize and are confusing at first.

Typically when we decompose something, we do it into a simpler form but here seemingly a more complex form is being employed. A simpler quantity, a cosine wave is now decomposed into two complex functions. But the net result is that it will make Fourier analysis simpler. We will go from simplicity to complexity and then to simplicity again.

Let's take this sinusoid which has a phase term to complicate things and present it in complex form.

$$\begin{aligned} x(t) &= A\cos(\omega t + \theta) \\ &= \frac{A}{2}e^{j(\omega t+\theta)} + \frac{A}{2}e^{-j(\omega t+\theta)} \\ &= \frac{A}{2}e^{j\omega t}e^{j\theta} + \frac{A}{2}e^{-j\omega t}e^{-j\theta} \end{aligned}$$

In the last row, we separated the exponential into its powers. If we expand this expression into trigonometric domain using Euler's equation, we see that indeed we do get back the trigonometric cosine wave we started with.

$$\begin{aligned} &= \frac{A}{2}\Big(\cos(\omega t + \theta) + \cancel{j\sin(\omega t + \theta)} + \cos(\omega t + \theta) - \cancel{j\sin(\omega t + \theta)}\Big) \\ &= A\cos(\omega t + \theta) \end{aligned}$$

Fourier series representation using complex exponentials

In Chapter 1, we used used trigonometric harmonics (the sine and cosine) as a basis set to develop the Fourier series representation. The target signal was "mapped" on to a set of sinusoidal harmonics, such as these based on fundamental frequency of ω_0.

$$S = \Big[\sin\omega_0 t, \cos\omega_0 t, \sin 2\omega_0 t, \cos 2\omega_0 t, \dots\Big]$$

What is a complex exponential? Well, we can think of it as a little suitcase packed with two waves, a sine and a cosine of the same frequency. Hence it allows us to write the Fourier series in a more compact form, with one CE representing both a sine and a cosine (with dreaded j thrown in). A set of complex exponentials given by set S, can then be used alternately as a basis set for creating a complex (but preferred) form of the Fourier series.

$$S = \Big[\dots, e^{-3\omega_0 t}, e^{-2\omega_0 t}, e^{-1\omega_0 t}, 1, e^{1\omega_0 t}, \dots, e^{m\omega_0 t}, \dots\Big]$$

These complex exponentials also form an orthogonal set, making them easy to separate from each other in an arbitrary signal. This is the main reason why we pick orthogonal signals to represent something. Just as our 3-D world is defined along three orthogonal axes, X, Y and Z, our signals can be similarly projected on a K-dimensional orthogonal set.

Recall that the Fourier series is a sum of weighted sinusoids. By weighted we mean that each sinusoid has its own amplitude and starting phase. The time is continuous but frequency resolution is not. Frequency takes on discrete harmonic values. If the fundamental frequency is ω, then each ω_k is an integer multiple of ω or $k\omega$ hence is discrete no matter how large k gets. The "distance" between each harmonic remains the same, ω.

$$f(t) = a_0 + \sum_{k=1}^{K} a_k \cos(\omega_k t) + \sum_{k=1}^{K} b_b \sin(\omega_k t) \tag{2.10}$$

The coefficients a_0, a_k and b_k (which we call the trigonometric coefficients) are calculated by (from Chapter 1)

$$\begin{aligned} a_0 &= \frac{1}{T_0}\int_0^{T_0} f(t)dt \\ a_k &= \frac{2}{T_0}\int_0^{T_0} f(t)\cos(k\omega t)dt \\ b_k &= \frac{2}{T_0}\int_0^{T_0} f(t)\sin(k\omega t)dt \end{aligned} \tag{2.11}$$

The presence of the integral tells us that time is continuous. Now substitute Eq. (2.3), and Eq. (2.4) as the definition of sine and cosine into Eq. (2.10), to get

$$f(t) = a_0 + \sum_{k=1}^{K} \frac{a_k}{2}\left(e^{jk\omega t} + e^{-jk\omega t}\right) + \sum_{k=1}^{K} \frac{b_k}{2j}\left(e^{jk\omega t} - e^{-jk\omega t}\right) \tag{2.12}$$

Make the same substitution in Eq. (2.11).

$$a_k = \frac{2}{T_0}\int_0^{T_0} f(t)\frac{1}{2}\left(e^{jk\omega t} + e^{-jk\omega t}\right)dt \tag{2.13}$$

$$b_k = \frac{2}{T_0}\int_0^{T_0} f(t)\frac{1}{2j}\left(e^{jk\omega t} - e^{-jk\omega t}\right)dt \tag{2.14}$$

Rearranging Eq. (2.12) so that each exponential is separated, we get

$$f(t) = a_0 + \sum_{k=1}^{\infty} \frac{1}{2}(a_k - jb_k)e^{jk\omega t} + \sum_{k=1}^{\infty} \frac{1}{2}(a_k + jb_k)e^{-jk\omega t} \tag{2.15}$$

The coefficients in Eq. (2.13) can also be expanded as follows.

$$a_k = \frac{1}{T_0}\int_0^{T_0} f(t)e^{jk\omega t}dt + \frac{1}{T_0}\int_0^{T_0} f(t)e^{-jk\omega t}dt \tag{2.16}$$

Look at this equation carefully. You see that the trigonometric coefficient is split into two parts now, one for each of the exponentials. To make the new coefficients concise, let's redefine them with capital letters as complex coefficients, A_k and B_k

$$A_k = \frac{1}{2}(a_k + jb_k) \tag{2.17}$$

$$B_k = \frac{1}{2}(a_k - jb_k) \tag{2.18}$$

Substituting these new definitions of the coefficients into Eq. (2.16), we get a much simpler representation.

$$\begin{aligned} f(t) &= a_0 + \sum_{k=1}^{\infty} A_k e^{jk\omega t} + \sum_{k=1}^{\infty} B_k e^{-jk\omega t} \\ \text{where} \quad A_k &= \frac{1}{T_0}\int_0^{T_0} f(t)e^{jk\omega t}dt \\ B_k &= \frac{1}{T_0}\int_0^{T_0} f(t)e^{-jk\omega t}dt \end{aligned} \tag{2.19}$$

It is clear from this equation that A_k can be thought of as the coefficient of the positive exponential and B_k the coefficient of the negative exponential. These coefficients are not the same as the ones we computed in the trigonometric form. They are complex combinations of the trigonometric coefficients a_k and b_k.

The term a_0 stands for the DC component. We generally do not like DC terms so we will remove it by expanding the range of the index k from 0 to ∞. Rewrite Eq. (2.19) as

$$\boxed{f(t) = \sum_{k=0}^{\infty} A_k e^{jk\omega t} + \sum_{k=0}^{\infty} B_k e^{-jk\omega t}} \tag{2.20}$$

The above equation can be simplified still further by extending the range of coefficients from $-\infty$ to ∞. We can do this by changing the sign of the index which was one-sided because we had included both positive and negative exponentials explicitly. Now both terms can be combined into one with a two-sided index to write a much more compact and elegant equation for the Fourier series. Now we do not need the negative exponential in the equation. The index takes care of that. And here is a much shorter equation for Fourier series in the complex domain.

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega}} \tag{2.21}$$

The coefficient C_k in Eq. (2.21) is given by

$$\boxed{C_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)e^{-jk\omega_0 t}dt} \tag{2.22}$$

C_k is of course equal to the trigonometric coefficients in this fashion.

$$\begin{aligned} C_k =& \frac{1}{2}(a_k - jb_k) \text{ if } k \geq 0 \\ =& \frac{1}{2}(a_k + jb_k) \text{ if } k < 0 \end{aligned}$$

The Eq. (2.22) is called the complex form of the Fourier series. It is rigorously related to the sinusoidal form. The magnitude calculated using the trigonometric form is exactly the same as the magnitude from this from. It is most used form of the Fourier series.

As explained in Chapter 1, for the trigonometric form, the index k is always positive and therefore the spectrum for the Fourier series using the trigonometric form is one-sided. The x-axis for the one-sided spectrum is plotted against frequency starting at a "positive" fundamental frequency. All k integer multiples of the fundamental frequency are positive as well. Because the index is positive, all frequencies are said to be positive.

If the two forms are equivalent, then how do we get a negative frequency in the complex form of the Fourier series? Does the frequency actually become negative when we use the complex exponentials? This is a often asked question.

With complex formulation, the index k spans from $-\infty$ to $+\infty$. We start with the negative index, go through calculations of all negative exponent exponentials and then the positive ones. Note that at no point is the fundamental frequency ever negative. Hence it is not the frequency of the exponential that is negative but *just* the index. The exponential with the negative index k is different from the positive exponential in that the sign of the imaginary part is negative. We see nothing here that says that the frequency has suddenly become negative because of the exponential exponent is negative.

$$\begin{aligned} e^{+jk\omega t} &= \cos(k\omega t) + j\sin(k\omega t) \\ e^{-jk\omega t} &= \cos(k\omega t) - j\sin(k\omega t) \end{aligned} \tag{2.23}$$

We now equate this form with the trigonometric form which seemingly had only positive frequencies. To do that we look at what it takes to represent a sine and a cosine using complex exponentials.

$$\begin{aligned} \cos(k\omega t) &= \frac{1}{2}\left(e^{jk\omega t} + e^{-jk\omega t}\right) \\ \sin(k\omega t) &= \frac{1}{2j}\left(e^{jk\omega t} - e^{-jk\omega t}\right) \end{aligned} \tag{2.24}$$

We require both a negative-index exponential and a positive-index exponential for both the sines and cosines. Where index k is always positive on the left hand side of this equation, it is both negative and positive on the right side. This traps us into thinking that frequency has changed sign. Where in trigonometric form a positive index is enough to fully and completely represent the signal, in complex form it takes a double-sided index. The spectrum is plotting the product of the index and the frequency, $(k \ldots \omega)$ and not just the frequency, ω on the x-axis. But we very quickly lose sight of this fact. We start talking about positive and negative frequencies because we confuse the *range* of the index with the sense of the frequency.

In a double-sided spectrum we are using the word frequency as an alternate name for what we are really plotting, and what we are actually plotting is the index times the frequency. Calling it frequency gives us some intuitive comfort but then we have to worry about what a negative frequency means. When we say *negative frequency*, we have in fact unknowingly converted a complex idea into simple everyday language. Because of the plotting convention, the negative index is oft-forgotten and the axis is referred to as the frequency axis, spanning both positive and negative domains. In this book, we maintain that there is no such thing as a negative frequency. The idea comes from confusion caused by what the x-axis represents.

The complex coefficient values are one-half of what they are calculated in the trigonometric domain. Now here students of the subject make up another story that this is because the frequency is being split into two parts, a negative part and positive part with each getting half the coefficient. This is how most books try to explain the conundrum of positive and negative frequency in relation to Fourier analysis. But they are just trying to explain a plotting convention. The real story is that we are not plotting frequency on the x-axis but the term $\pm k\omega$ which is often just referred to of as frequency.

The reason the values are split in half can be explained intuitively. We have let the index k go from $-\infty$ to $+\infty$, so now each frequency is multiplied by both a positive k and a negative k. However, in reality, each frequency has only a finite energy, so to make it all work out, the amplitude of this frequency is split between these two index.

There are certain things that are defined only as positive quantities, such as volume, mass, age, etc. and frequency as a physical property is one of those things. However the word frequency can indeed and is often used in a complex sense to include several parameters of a wave.

Example 2.1. Compute complex coefficients of a cosine wave.

$$\begin{aligned} f(t) &= A\cos(\omega_0 t) \\ &= \frac{A}{2}e^{j(k=1)\omega_0 t} + \frac{A}{2}e^{j(k=-1)\omega_0 t} \end{aligned} \tag{2.25}$$

This example is so simple that we can easily deduce the trigonometric coefficients just by looking at the expression. In fact the equation is itself the perfect representation. The complex coefficients are of magnitude $A/2$ located at $k = 1$ and $k = -1$. We plot the trigonometric coefficients, C_k, in Fig. 2.5 as the single-sided spectrum as well as the exponential coefficients, C_k, as the double sided form. The x-axis variable is $k\omega$. Since ω is a constant, we are really plotting, k, the index. Note it is not the frequency that is negative but the harmonic index k. However in a typical plot, the x-axis is labeled as frequency. In these plots, we have labeled it specifically as what it is, the term $k\omega$.

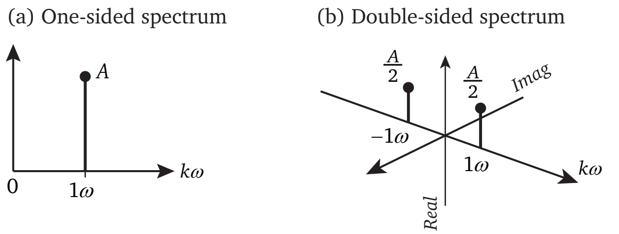

Figure 2.5: The spectrum of $A\cos\omega t$. The trigonometric form gives us a one-sided spectrum with one component located the ω. The complex form shows two components of half the total amplitude in the real plane. Both components have positive amplitudes. The total energy in both forms is the same.

Example 2.2. Compute complex coefficients of a sine wave

$$\begin{aligned} f(t) &= A\sin\omega_0 t \\ &= \frac{A}{2j}e^{j(k=1)\omega_0 t} - \frac{A}{2j}e^{j(k=-1)\omega_0 t} \end{aligned} \tag{2.26}$$

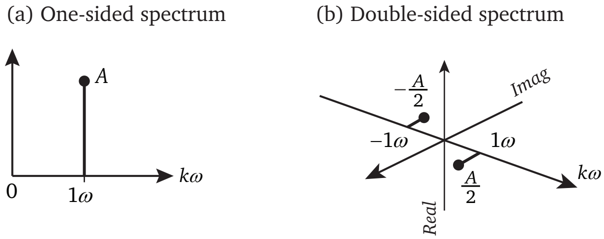

Figure 2.6: The spectrum of $A\sin\omega t$. The trigonometric form gives us a one-sided spectrum with one component located the ω. The complex form shows two components of half the total amplitude of opposite signs in the imaginary plane.

This example is just the same as the cosine example. The single-sided spectrum is easy. It is simply a harmonic of magnitude A and located at $k = 1$ just as it is for the cosine wave in Ex. 2.1. In Eq. (2.26), we write the complex form with the amplitudes of the two complex exponentials of $A/2$ and $-A/2$ located at $k = \pm 1$. However there is a j in the denominator. What to do with this? The presence of j tells us that the coefficients are on the Imaginary axis, so they are to be plotted on the Imaginary plane, right-angle to the plane on which a cosine lies. Drawn in 2-D form it has no computational effect, only that the vertical axis is called the Imaginary axis. But when we have both cosines and sine waves present in a signal, the coefficients of these two have to be combined not linearly but as a vector sum as seen in Fig. 2.7. Why? Because the harmonics are orthogonal to each other. When plotting the magnitude, it no longer falls in purely *Real* or *Imaginary* planes so in this case, we call the vertical axis, just the magnitude. There is a bit of terminology sloppiness here, often that is what makes signal processing so confusing.

Example 2.3. Compute the coefficients of $f(t) = A(\cos\omega t + \sin\omega t)$. We can write this wave as

$$f(t) = \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t} + \frac{A}{2j}e^{j\omega t} - \frac{A}{2j}e^{-j\omega t} \tag{2.27}$$

We can pick out the trigonometric coefficients from the first equation. It is simply A for the cosine and A for sine with magnitude equal to square root of $\sqrt{2}A$ located at $\omega = 1$. We get the complex coefficients by looking at the coefficients of the two exponentials in Eq. (2.27).

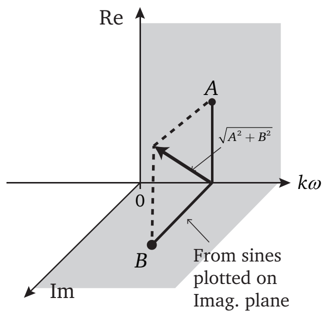

Figure 2.7: Magnitude of the resultant vector if a signal contains both a sine and a cosine.

The $e^{j\omega t}$ exponential has two coefficients, at 90 degrees to each other, each of magnitude $A/2$. The vector sum of these is $A/\sqrt{2}$. Same for the negative exponential, except the amplitude contribution from the sine is negative. However, the vector sum or the magnitude for both is the same and always positive. This is shown in Fig. 2.8(c) drawn in a more conventional style showing only the vector sum. Note that the total energy (sum of magnitudes) of the one-sided spectrum is exactly the same as that of complex.

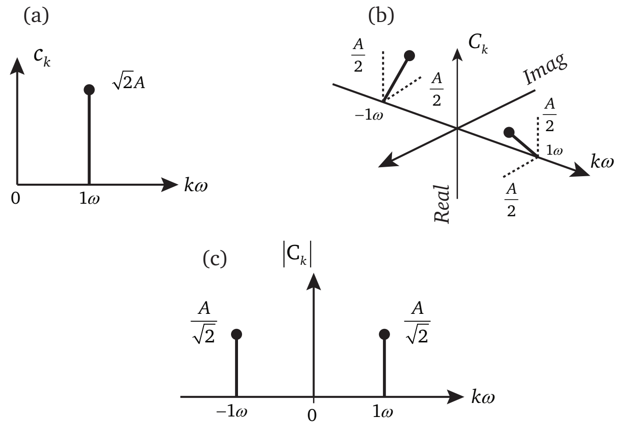

Figure 2.8: Amplitude spectrum of $A\sin\omega t + A\cos\omega t$.

Example 2.4. Compute coefficients of the complex signal $f(t) = A\cos\omega t + j\sin\omega t$. We can write this as

$$f(t) = \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t} + \frac{A}{2}e^{j\omega t} - \frac{A}{2}e^{-j\omega t} = Ae^{j\omega t}$$

Now here we see something different. The coefficients from sine and cosine for the negative exponential cancel out. On the positive side, the contribution from sine and cosine are coincident and add. So we see a single value at the positive index of $k = 1$ only. For this signal both the single and double-sided spectrum are identical. This is a surprising and perhaps a counter-intuitive result.

Important observation: *Only real signals have symmetrical spectrum about the origin. Complex signals do not.*

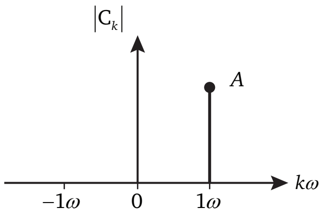

Figure 2.9: Double-sided spectrum of $A\cos\omega t + jA\sin\omega t$.

Example 2.5. Compute the coefficients of a constant signal, $f(t) = A$.

We can write this function $x(t) = A$ or as an exponential of zero frequency.

$$\begin{aligned} f(t) &= A\cos(\omega = 0)t \\ &= \frac{A}{2}e^{j(\omega=0)t} + \frac{A}{2}e^{-j(\omega=0)t} \\ &= A. \end{aligned}$$

Thee trigonometric coefficient is $= A$ at $\omega = 0$. For the complex representation we get two complex coefficients, both of amplitude $A/2$ and $A/2$ but both at $k = 0$ so their sum is A which is exactly the same as in the trigonometric representation. The function $f(t)$, a constant is a non-changing function of time and we classify it as a DC signal. The DC component, if any, always shows up at the origin for this reason. The single and double sided spectrum here are same as well. This signal has only one component at $\omega = 0$, hence it has only one, coefficient a_0. All the others are zero. So that is what we are seeing here in Fig. 2.10. Just the a_0 coefficient plotted.

Important observation: *A component at zero frequency means that the signal is not zero-mean.*

Example 2.6. Compute coefficients of $x(t) = 2\cos^2(\omega t)$.

We can express this function in complex form as

$$\begin{aligned} x(t) &= 2\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right)^2 \\ &= 1 + \frac{1}{2}e^{j2\omega t} + \frac{1}{2}e^{-j2\omega t} \end{aligned}$$

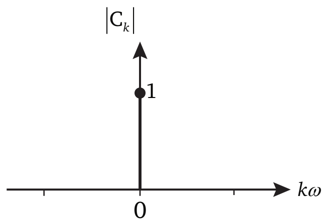

Figure 2.10: Double-sided spectrum of a constant signal of amplitude A. A constant signal is same as a DC value, hence its spectrum always appears as an impulse at the origin.

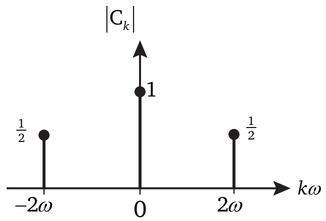

Figure 2.11: Double-sided amplitude spectrum of $2\cos^2(\omega t)$.

Observation: *A squared signal by definition is always positive so the spectrum has a zero-frequency component in the center.*

Example 2.7. Compute the coefficients of $x(t) = 2\cos(\omega t)\cos(2\omega t)$.

We can express this signal in complex form by making use of this trigonometric identity: $cos(a+b) = cos(B)cos(A) + sin(A)sin(B)$.

$$\begin{aligned} x(t) &= \cos(\omega t) + \cos(3\omega t) \\ &= \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t} + \frac{1}{2}e^{j3\omega t} + \frac{1}{2}e^{-j3\omega t} \end{aligned}$$

Of course doing this in trigonometric form would have been just as easy. But that is not always true. We draw the spectrum as in Fig. 2.12.

Here too, we can apply the thinking process we saw in Chapter 1. This signal has two components at $\omega = 1,\ 3$, hence it has only two (four) coefficients, coefficient $a_{\pm 1}$ and $a_{\pm 3}$. All the others are zero. So that is what we are seeing in Fig. 2.12. Just those four coefficient plotted, with half the amplitude for each from the time domain equation.

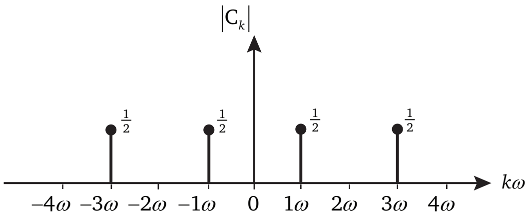

Figure 2.12: Double-sided amplitude spectrum of $2\cos(\omega t)\cos(2\omega t)$.

Example 2.8. Compute the complex coefficients of this real signal.

$$f(t) = \sin(2\pi t) + .8\cos(8\pi t) + .3\sin(14\pi t)$$

The signal has three harmonics, at $k = \pm 1, \pm 4, \pm 7$. We can write this equation in complex form as

$$\begin{aligned} f(t) = &\frac{1}{2}e^{j2\pi(k=1)t} + \frac{1}{2}e^{j2\pi(k=-1)t} \\ &+ \frac{0.8}{2}e^{j2\pi(k=4)t} + \frac{0.8}{2j}e^{j2\pi(k=-4)t} \\ &+ \frac{0.3}{2j}e^{j2\pi(k=7)t} + \frac{0.3}{2}e^{j2\pi(k=-7)t} \end{aligned}$$

Here we have contributions from both sine and cosine at $k = 1$, so these have to be vector summed. The contributions at $k = 2$ comes only from a cosine and at $k = 7$ only from a sine. Note we plot these on the same line at full magnitude as if j does not exist in the equation. (We will drop mentioning the index k and call it frequency to be consistent with common usage. However, note that it is this sloppiness in terms that causes us to question our sanity and start asking: what is a negative frequency?) This signal has three components

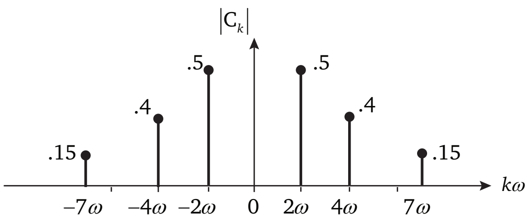

Figure 2.13: Two-sided magnitude spectrum.

at $\omega = 1$, 4 and 7, hence it has only three (six) coefficients, coefficient $a_{\pm 1}$, $a_{\pm 4}$ and $a_{\pm 7}$. All the others are zero. So that is what we are seeing here in Fig. 2.13. Just those three coefficients plotted on each side, with half the amplitudes from the time domain equation.

Example 2.9. Compute the complex coefficients of this real signal with phase terms. Then compute its Power spectrum.

$$x(t) = 3 + 6\cos(4\pi t + 2) + j\sin(4\pi t + 3) - j6\sin(10\pi t + 1.5)$$

We convert this to the complex form as

$$\begin{aligned} x(t) &= 3 + \left(3e^{j4\pi t}e^{2} + 3e^{-j4\pi t}e^{-j2}\right) + \left(2e^{j4\pi t}e^{3} - 2e^{-j4\pi t}e^{-j3}\right) + \left(3e^{j10\pi t}e^{j1.5} - 3e^{-j10\pi t}e^{-j1.5}\right) \\ &= 3 + e^{j4\pi t}\left(3e^{j2} + 2e^{j3}\right) + e^{-j4\pi t}\left(3e^{-j2} - 2e^{-j3}\right) + 3e^{j10\pi t}\left(e^{j1.5}\right) + 3e^{-j10\pi t}\left(e^{-j1.5}\right) \end{aligned}$$

The magnitudes of the exponentials come from the phasors in parenthesis. To add them we need to convert them first to rectangular form as follows. (See Appendix A). The CE $e^{j4\pi t}$ has the following coefficients.

$$\begin{aligned} e^{j4\pi t} &\rightarrow \left(3e^{j2} + 2e^{j3}\right) \\ \Rightarrow \left|3e^{j2} + 2e^{j3}\right| &= \sqrt{\left(3\cos(2) + 2\cos(3)\right)^2 + \left(3\sin(2) + 2\sin(3)\right)^2} \\ &= 4.414. \end{aligned}$$

Similarly, the coefficient of the negative exponential is

$$\begin{aligned} e^{-j4\pi t} &\rightarrow \left(3e^{-j2} - 2e^{-j3}\right) \\ \Rightarrow \left|3e^{-j2} - 2e^{-j3}\right| &= \sqrt{\left(3\cos(2) - 2\cos(3)\right)^2 + \left(3\sin(2) + 2\sin(3)\right)^2} \\ &= 3.098. \end{aligned}$$

We draw the spectrum in Fig. 2.14 and note that the spectrum is not symmetric because the signal is complex.

Important Observation: Most signals we work with are complex hence their spectrum are rarely symmetrical.

Power spectrum

You may be familiar with this expression of power from a circuits class. Power is equal to:

$$P = V^2/R$$

Here V is the voltage or the amplitude of the signal and R the resistance. Let's just assume that R is equal to 1.0, in this case, the normalized power is equal to the voltage squared. If we square the peak voltage, we get peak power, and if we square the mean voltage, we

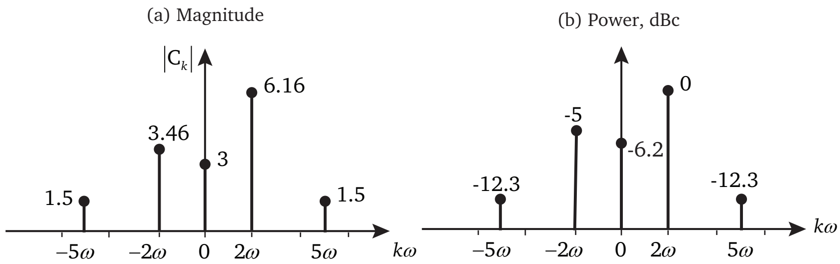

Figure 2.14: (a) Two-sided Magnitude spectrum of a complex signal, (b) its power spectrum computed by squaring each component and converting to dBs.

get the average power. This idea is in fact exactly the same as Parseval's Theorem, which states that the power in a particular harmonic is equal to the square of its amplitude or the coefficient. So for this particular example, to obtain the Power spectrum, we just square each amplitude, convert it into dBs and then normalize for maximum power. The result is shown in Fig. 2.14(b).

Now a difficult but a very important example, a periodic signal of repeating square pulses.

Example 2.10. Compute the Fourier coefficients of the following periodic square wave. The square wave is of amplitude 1 that lasts τ seconds and repeats every T seconds. (Note that τ and T are different and independent.) First we compute the coefficients for a general case

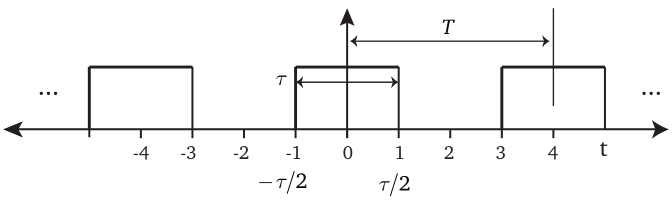

Figure 2.15: A Square wave of period T and duty cycle $2/T$.

for pulse time equal to τ sec. and repeat time equal to T sec. The term τ/T is called the duty cycle of the wave.

$$C_k = \frac{1}{T}\int_{-\tau/2}^{\tau/2} 1 \cdot e^{-j2k\pi f_0 t} dt$$

Note that outside of $-\tau/2 < t < \tau/2$, the function is zero. This a pretty easy integral given by

$$\begin{aligned} C_k &= \frac{1}{T}\frac{e^{-(j2\pi k/T)t}}{-j2\pi k/T}\bigg|_{-\tau/2}^{\tau/2} \\ &= \frac{1}{T}\left(\frac{e^{(j2\pi k/T)\frac{\tau}{2}} - e^{(-j2\pi k/T)\frac{\tau}{2}}}{j2}\right)\frac{1}{\pi k/T} \\ &= \frac{\tau}{T}\frac{\sin(k\pi\tau/T)}{k\pi(\tau/T)} \end{aligned}$$

Replacing the duty cycle term τ with r, the equation becomes fairly easy to understand.

$$C_k = r\frac{\sin(k\pi r)}{k\pi r} = r\,\text{sinc}(k\pi r).$$

The duty cycle of this signal is equal to

$$r = \frac{\tau}{T} = \frac{1}{2}$$

Substituting that in the above equation, we can write the expression for the coefficients of this signal as

$$C_k = \frac{1}{2}\text{sinc}(k\pi/2) \tag{2.28}$$

This is the *sinc* function. It comes up so often in signal processing that it is probably the second most important equation in DSP after the Euler's equation.

Now we can plot the coefficients of the repeating continuous-time square pulse coefficients for various duty cycles. Note how the peakedness of the main lobe changes in opposite fashion to the duty cycle. A narrow pulse relative to the period in Fig. 2.16(a), has a shallower frequency response than one that takes up more of the period. The zero-crossings occur at inverse of the duty cycle. For $r = 0.5$, the zero-crossing occurs at $k = 2$, for $r = .25$, the crossing is at $k = 4$ and for $r = .75$, the crossing occurs at $n = 1.33$. At $r = 1.0$, the pulse would be a flat line and it will have an impulse at its frequency response. For very small r, the pulse is delta function-like and the response will go to a flat line. Note the usage of words, frequency response. This is just another term for what we have been calling the spectrum.

Although the equation for this function is fairly easy, it takes a while to develop intuitive feeling. We cannot over emphasize the importance of this signal and you ought to spend some time playing around with the parameters to understand the effect. We will of course keep coming back to it in the next chapters.

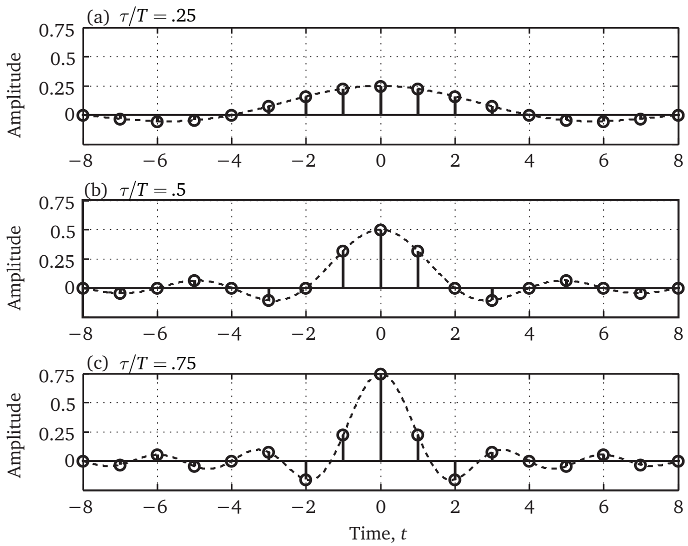

Figure 2.16: Fourier coefficients of a square pulse.
Note that as the pulse gets narrower in time, i.e. its duty cycle is small, then its frequency response is shallow. But as the duty cycle increases, such as in (c), the response is becoming peaked.

In this chapter, we covered the complex form of the Fourier series as a prelude to the next topic, the Fourier Transform. We see that even though the time domain function is a continuous periodic function, the Fourier series coefficients and hence the spectrum developed is discrete. In Chapter 3 we will see how that affects the Fourier analysis.

Summary of Chapter 2

In this chapter we look at the complex exponential as a concise way of representing the sinusoids. They not only make the Fourier representation shorter to type, they also make the math easier. The complex exponential is nothing but a three-dimensional sinusoid. The spectrum of the Fourier series when using the complex form is double-sided, which means that the frequency index spans from $-\infty$ to $+\infty$.

Terms used in this chapter:

- **Euler's equation**
- **Continuous-time Complex exponential**, $e^{j\omega t}$ of frequency ω.
- **Complex coefficients of Fourier series**, C_k
- **Double-sided spectrum**

1. The sine and cosine can be represented by the complex exponentials. We use the following expressions to represent them in Fourier series to obtain a complex form of the Fourier series equation.

$$\cos(k\omega t) = \frac{1}{2}\left(e^{jk\omega t} + e^{-jk\omega t}\right)$$
$$\sin(k\omega t) = \frac{1}{2j}\left(e^{jk\omega t} - e^{-jk\omega t}\right)$$

2. To represent a periodic signal using the complex exponentials requires a double sided harmonic index k, unlike the trigonometric case where the harmonic index is positive.
3. The harmonic index extends from $-K \leq k \leq +K$. K can span from -∞ to $+\infty$.
4. The x-axis now represent values from $-K\omega_0 \leq k\omega_0 \leq +K\omega_0$ and this is often read as representing negative frequency when in fact it is the index that is negative.
5. The Fourier series coefficients instead of being of three types can now be represented by a single equation.

$$C_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)e^{-jk\omega_0 t}dt$$

6. The fundamental properties remain the same, the time in this representation is continuous and frequency is discrete with index k, which is an integer.
7. The amplitude spectrum obtained from the complex representation looks different from the one-sided spectrum. We call this spectrum a two-sided spectrum. The amplitude value for a particular harmonic is now split in half for the positive index (so called positive frequency) and half for the negative index (so called negative frequency). The

0-th component however remains the same. This conserves energy and makes both forms equivalent.

Questions

1. What is a complex exponential?
2. What is the expression for a sine in the complex from?
3. What is the value of $e^{-j\pi}$, $e^{j2\pi}$, $e^{-j\frac{3}{4}\pi}$.
4. Why is this equation true? Give your answer in words.

$$e^{j\pi}+1=0$$

5. What is the difference between two complex exponential of the same exponent but different signs, such as $e^{j\omega_k t}$ and $e^{-j\omega_k t}$. If we add these two signals, what do we get in the complex domain?
6. What dimensional space is required to plot a complex exponential?
7. The term phasor is often used in relation to complex exponentials, what is it?
8. If you plot a sinusoid plus its shifted version, $\sin(2\omega)+\sin(2\omega+\phi_0)$, what is the phase of the new signal?
9. Represent this sinusoid as a complex exponential: $\cos\left(\omega t+\frac{\pi}{4}\right)$.
10. What is the relationship of Taylor series to a sinusoid?
11. What is the advantage of using complex exponential as a basis set instead of sines and cosines?
12. Given these CEs, give their expression in the Euler from: e^{-j3}, $(e^{-j4}+e^{-j2})$, $e^{-j\frac{\pi}{2}t}$
13. How would you express phase in the complex exponential form?
14. Write the CE form of these signals:

$$\sin(7\pi t+\frac{\pi}{4})\cos(7\pi t+\frac{\pi}{4})-\sin(7\pi t-\frac{\pi}{4})\cos(3\pi t+\frac{\pi}{2})$$

15. What is the magnitude of this complex signal, $2e^{-5jt}$?
16. How do we transmit a complex signal? What does it look like?
17. What does division by j mean?
18. What does multiplication by j mean?
19. What is the magnitude and phase of these signals:

$$\big(\sin(\omega t)-\cos(\omega t)\big);\big(\frac{1}{2}\cos(\omega t)-\sin(\omega t)\big);\big(2\cos(\omega t)-\frac{1}{2}\sin(\omega t)\big)$$

20. What is a single-sided spectrum? What does it represent?
21. Given the amplitude spectrum, how would you compute the power spectrum?

22. What is two-sided spectrum of these signals?

$$f(t) = -\cos(2\pi t) + \cos(9\pi t) + \sin(12\pi t)$$
$$f(t) = 2\cos(9\pi t) + 2\cos(18\pi t)$$

23. When plotting a two sided spectrum, what does the x-axis represent? Is it frequency?
24. If you are given the real and imaginary components of a signal, how do you compute the phase? Is phase changing with time or frequency?
25. For a complex signal, both real and imaginary signals can have non-zero phase, so what is the phase of a complex signal? How is it different from the phases of the components?
26. What is the relationship of the trigonometric coefficients to the complex coefficients?
27. Why the spectrum of a complex signal is always one-sided?
28. If we have a periodic signal of square pulses with a duty cycle of .1. How much wider is its spectrum as compared to a pulse that has a duty cycle of .5?
29. What happens to the Magnitude spectrum if phase of the signal changes?

Appendix A: A little bit about complex numbers

We can use complex numbers to denote quantities that have more than one parameter associated with them. A point in a plane is one example. It has a y coordinate and a x coordinate. Another example is a sine wave, it has a frequency and a phase. The two parts of a complex number are denoted by the terms Real and Imaginary, but the Imaginary part is just as real as the Real part. Both are equally important because they are needed to nail down a physical signal.

The signals traveling through air are real signals and it is only the processing that is done in the complex domain. There is a very real analogy that will make this clear. When you hear a sound, the processing is done by our brains with two orthogonally placed receivers, the ears. The ears hear the sound with slightly different phase and time delay. The received signal by the two ears is different and from this our brains can derive fair amount of information about the direction, amplitude and frequency of the sound. So although yes, most signals are real, the processing is often done in complex plane if we are to drive maximum information.

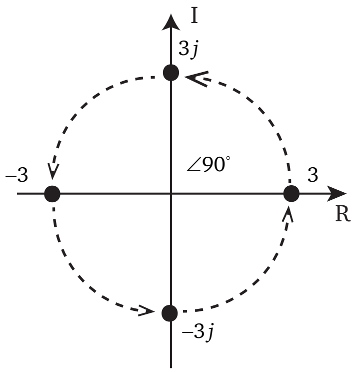

Figure A.1: Multiplication by j shifts the location $0f$ a point on a plane by $90°$.

The concept of complex numbers starts with real numbers as a point on a line. Multiplication of a number by -1 rotates that point 180° about the origin on the number line. If a point is 3, then multiplication by -1 makes it -3 and it is now located 180° from $+3$ on the number line. Multiplication by -1 can be seen as 180° shift. Multiplying this rotated number again by -1, gives the original number back, which is to say by adding another 180° shift. So multiplication by $(-1)^2$ results in a 360° shift. What do we have to do to shift a number off the line, say by 90°? This is where j comes in. Multiply 3 by j, so it becomes $3j$. Where do we plot it now? Herein lies our answer to what multiplication with j does. Multiplication by j moves the point off the line.

Question: What does division by j mean?

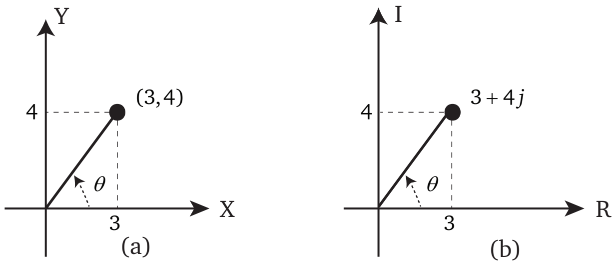

Figure A.2: a. A point is space on a Cartesian diagram. b. Plotting a complex function on a complex plane.

Answer: It is same as multiplying by $-j$.

$$\frac{x}{j} \times \frac{j}{j} = \frac{jx}{-1} = -jx.$$

This is essentially the concept of complex numbers. Complex numbers often thought of as "complicated numbers" follow all of the common rules of mathematics. Perhaps a better name for complex numbers would have been 2D numbers. To further complicate matters, the axes, which were called X and Y in Cartesian mathematics are now called respectively *Real* and *Imaginary*. Why so? Is the quantity $j3$ any less *real* than 3? This semantic confusion is the unfortunate result of the naming convention of complex numbers and helps to make them confusing, complicated and of course complex.

Now let's compare how a number is represented in the complex plane.

Plot a complex number, $3 + j4$. In a Cartesian plot we have the usual $X - Y$ axes and we write this number as $(3,4)$ indicating 3 units on the X-axis and 4 units in the Y-axis. We can represent this number in a complex plane in two ways. One form is called the rectangular form and is given as

$$z = x + jy$$

The part with the j is called the imaginary part (although of course it is a real number) and the one without is called the real part. Here 3 is the Real part of z and 4 is the Imaginary part. Both are real numbers of course. Note when we refer to the imaginary part, we do not include j. The symbol j is there to remind you that this part (the imaginary part) lies on a different axis.

$$Re(z) = x$$
$$Im(z) = y$$

Alternate form of a complex number is the **polar form**.

$$z = M\angle\theta$$

where M is the magnitude and θ its angle with the real axis.

The polar form which looks like a vector and in essence it is, is called a ***Phasor*** in signal processing. This idea comes from circuit analysis and is very useful is that realm. We also use it in signal processing but it seems to cause some conceptual difficulty. Mainly because, unlike in circuit analysis, in signal processing time is important. We are interested in signals in time domain and the phasor which is a time-less concept is confusing. The phase as the term is used in signal processing is kind of the initial value of phase, where it is an angle in vector terminology.

Question: If $z = Ae^{j\omega t}$ then what is its rectangular form?

Answer: $z = A\cos\omega t + jA\sin\omega t$. We just substituted the Euler's equation for the complex exponential $e^{j\omega t}$. Think of $e^{j\omega t}$ as a shorthand functional notation for the expression $\cos\omega t + j\sin\omega t$ .The real and imaginary parts of z are given by

$$Re(z) = A\cos\omega t$$
$$Im(z) = A\sin\omega t.$$

Converting forms

Rule:

1. Given a rectangular form $z = x + jy$ then its polar form is equal to

$$M\angle\theta = \begin{cases} \sqrt{x^2+y^2}\angle\tan^{-1}\frac{y}{x} & \text{if } x \geq 0 \\ \sqrt{x^2+y^2}\angle\left(tan^{-1}\frac{y}{x}+\pi\right) & \text{if } x < 0 \end{cases}$$

2. Given a polar form $M\angle\theta$ then its rectangular form is given by

$$x + jy = M\cos\theta + jM\sin\theta$$

Example 2.11. Convert $z = 5\angle.927$ to rectangular form

$$Re(z) = 5\cos(.927) = 3$$
$$Im(z) = 5\sin(.927) = 4$$
$$\Rightarrow z = 3 + j4$$

Example 2.12. Convert $z = -1 - j$ to polar form

$$\begin{aligned} M &= \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} \\ \theta &= \arctan\frac{y}{x} + \pi, \quad \text{since } x < 0 \\ &= \arctan\frac{-1}{-1} + \pi = \frac{3}{4}\pi \\ \Rightarrow z &= \sqrt{2}\angle\frac{3}{4}\pi. \end{aligned}$$

Example 2.13. Convert $z = 1 + j$ to polar form

$$\begin{aligned} M &= \sqrt{(1)^2 + (1)^2} = \sqrt{2} \\ \theta &= \arctan\frac{y}{x}, \quad \text{since } x > 0 \\ &= \arctan\frac{1}{1} \\ &= \frac{1}{4}\pi \\ \Rightarrow z &= \sqrt{2}\angle\pi/4. \end{aligned}$$

Adding and Multiplying

Add in rectangular form, multiply in polar. Its easier this way. Rule

1. Given $z_1 = a + jb$ and $z_2 = c + jd$ then $z_1 + z_2 = (a + c) + j(b + d)$.
2. Given $z_1 = M_1\angle\theta_1$ and $z_2 = M_2\angle\theta_2$, then $z_1 \cdot z_2 = M_1M_2\angle(\theta_1 + \theta_2)$.

Example 2.14. Add $z_2 = \sqrt{2}\angle.785$ and $z_2 = 5\angle.927$.

Convert both to rectangular form

$$\begin{aligned} z_1 &= 1 + j \text{ and } z_2 = 3 + j4 \\ \Rightarrow z_3 &= (1 + 3) + j(1 + 4) = 4 + j5. \end{aligned}$$

Example 2.15. Multiply $z_1 = 1 + j$ and $z_2 = 3 + j4$.

First convert to polar form and then multiply. Although multiplying these two complex numbers in rectangular format looks easy, in general that is not the case. Polar form is better for multiplication and division.

$$z_1 \cdot z_2 = \sqrt{2}\angle.785 \times 5\angle.927 = 5\sqrt{2}\angle1.71$$

Example 2.16. Divide $z_1 = 1 + j$ and $z_2 = 3 + j4$.

$$\frac{z_1}{z_2} = \frac{\sqrt{2}\angle.785}{5\angle.927} = \frac{5}{\sqrt{2}}\angle.142$$

Conjugation

The conjugate for a complex number z, is given by $z^* = x - jy$. For a complex exponential $e^{j\omega t}$ is the complex conjugate of $e^{-j\omega t}$. In polar format the complex conjugate is same phasor but rotating in the opposite direction.

Rule: If $z = M\angle\theta$, then $z^* = M\angle - \theta$.

Useful properties of complex conjugates

$$|z|^2 = zz^*$$

This relationship is used to compute the power of the signal. The magnitude of the signal can be computed by half the sum of the signal and its complex conjugate. Note the imaginary part cancels out in this sum.

$$|z| = \frac{1}{2}\big(z + z^*\big).$$