Intuitive Guide to Fourier Analysis

Charan Langton
Victor Levin

Much of this book relies on math developed by important persons in the field over the last 200 years. When known or possible, the authors have given the credit due. We relied on many books and articles and consulted many articles on the internet and often many of these provided no name for credits. In this case, we are grateful to all who make the knowledge available free for all on the internet.

The publisher offers discounts on this book when ordered in quantity for bulk purchase or special sales. We can also make available on special or electronic version applicable to your business goals, such as training, marketing and branding issues. For more information, please contact us.

mntcastle@comcast.net

Website for this book: complextoreal.com/fftbook

ISBN- 13: 978-0-913063-26-2

4 | Continuous-time Fourier Transform (CTFT) of aperiodic and periodic signals

Harry Nyquist
February 7, 1889 - April 4, 1976

Harry Nyquist, was a Swedish born American electronic engineer who made important contributions to communication theory. He entered the University of North Dakota in 1912 and received B.S. and M.S. degrees in electrical engineering in 1914 and 1915, respectively. He received a Ph.D. in physics at Yale University in 1917. His early theoretical work on determining the bandwidth requirements for transmitting information laid the foundations for later advances by Claude Shannon, which led to the development of information theory. In particular, Nyquist determined that the number of independent pulses that could be put through a telegraph channel per unit time is limited to twice the bandwidth of the channel. This rule is essentially a dual of what is now known as the Nyquist-Shannon sampling theorem. – From Wikipedia

Applying Fourier series to aperiodic signals

In previous chapters we discussed the Fourier series as it applies to the representation of continuous and discrete signals. We discussed the concept of harmonic sinusoids as basis functions, first the trigonometric version of sinusoids and then the complex exponentials as a more compact form for representing a signal. The analysis signal is "projected" on to these basis signals, and the "quantity" of each basis function is interpreted as spectral content along a frequency line.

Fourier series discussions assume that the signal of interest is periodic. But a majority of signals we encounter in signal processing are not periodic. Even those that we think are periodic are not really so. Then we have many signals which are bunch of random bits with no pretense of *periodicity*. This is the real world of signals and Fourier series comes up short for these types of signals. This was of course noticed right away by the contemporaries of Fourier when he first published his ideas is 1822. Fourier series is great for periodic signals but how about stand-alone non-periodic, also called *aperiodic* signals like this one?

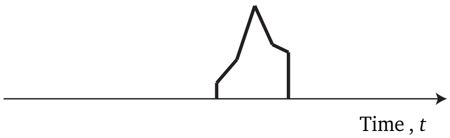

Figure 4.1: Can we compute the Fourier series coefficients of this aperiodic signal?

Taking some liberty with history, Fourier, I am sure was quite disappointed when he received a very unenthusiastic response to his work upon first publishing it. He was denied membership into the French Academy as the work was not considered rigorous enough. His friends and foes, who are now as famous as he is, (Laplace, Lagrange etc.) objected to his overreaching original conclusion about the Fourier series that it can represent *any* signal. They correctly guessed that series representation would not work universally, such as for exponential signals as well as for signals that are not periodic. Baron Fourier, disappointed but not discouraged, came back 20 years later with something even better, the *Fourier transform*. (If you are having a little bit of difficulty understanding all this on first reading, this is forgivable. Even Fourier took 20 years to develop it.)

Extending the period to infinity

In this chapter we will look at the mathematical trick Fourier used to modify the Fourier series such that it could be applied to signals that are not strictly periodic, or are transient. Take the signal in Fig. 4.1. Let's say that a little signal, as shown has been collected and the data shows no periodicity. Being engineers, we want to compute its spectrum using Fourier analysis, even though we have been told that the signal must be periodic. What to do?

Well, we can pretend that the signal in Fig. 4.1 is actually a periodic signal and we are only seeing one period. In Fig. 4.2(a) we show this signal repeating with a period of 5 seconds. Of course, we just made up that period. We truly have no idea what the period of this signal is, or if it even has one.

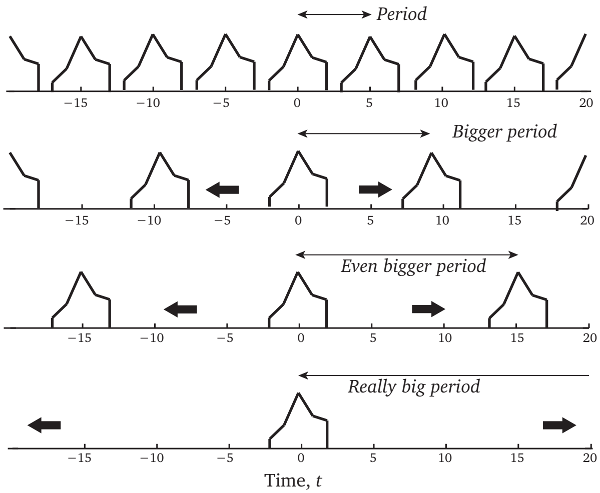

Figure 4.2: Going from periodic to aperiodic signal by extending the period.

Since 5 seconds is an assumed number anyway, let's just increase it some more by pushing these assumed copies out, increasing the time in-between. We can indeed keep doing this such that the zeros go on forever on each side and effectively the period becomes infinitely long. The signal is now just by itself with zeros extending to infinity on each side. We declare that this is now a periodic signal but with a period extending to ∞. We have turned an aperiodic signal into a periodic signal with this assumption. We can apply the Fourier analysis to this extended signal because it is *periodic*. Mathematically we have let the period T go to infinity so that the assumed copies of the little signal move so far apart that we see no hide nor hair of them. The single piece of the signal is then essentially part of a periodic signal which we can not see. The copies of the signal in Fig. 4.2(a) are of course fake, they are not really there. But with this assumption, the signal becomes periodic in a mathematical sense, and we can compute its Fourier series coefficients (FSC), by setting $T = \infty$.

This conceptual trick is needed because a signal must be periodic for Fourier series representation to be valid. When we have a signal that appears aperiodic, we can assume that the observed signal is part of a periodic series, although with such an infinitely long period that we don't even see any other data points. The Fourier series can be used for the spectral analysis of an aperiodic signals by this assumption.

In Fig. 4.3(a) we show a pulse train with period T_0. The Fourier series coefficients of the pulse train are plotted next to it (See Example 2.10). Note that as the pulses move further apart in (b) and (c), the spectral lines or the harmonics are moving closer together.

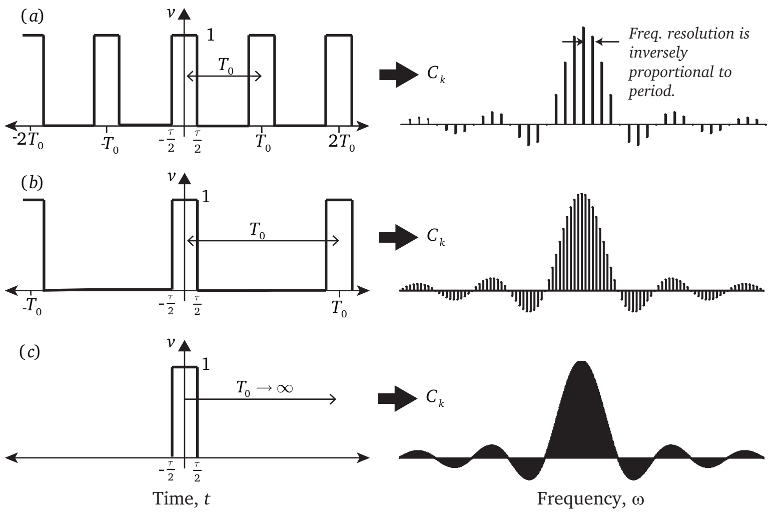

Figure 4.3: Take the pulse train in (a), as we increase the period, i.e., add more space between the pulses, the fundamental frequency gets smaller, which makes the spectral lines move closer together as in (c). In the limiting case, where the period goes to ∞, the spectrum would become continuous.

Continuous-Time Fourier Transform (CTFT)

It was probably this same observation that led Fourier to the Fourier transform. We can indeed apply Fourier series analysis to an aperiodic signal by assuming that the period of an arbitrary aperiodic signal is *very* long and hence we are seeing only one period of the signal. The aperiodic data represents one period of a presumed periodic signal, $\tilde{x}(t)$. But if the period is infinitely long, then the *fundamental frequency* defined as the inverse of the period becomes infinitely small. The harmonics are still integer multiples of this infinitely small fundamental frequency but they are so very close to each other that they approach a continuous function of frequency.

We will now go through the math to show how **Fourier transform** (FT) is directly derived from the Fourier series coefficients (FSC). Like much of the math in this book, it is not complicated, only confusing. However once you have clearly understood the concepts of fundamental frequency, period, and the harmonic frequencies, the rest gets easier.

After we discuss the Continuous-time Fourier Transform (CTFT), we will then look at the Discrete-Time Fourier Transform (DTFT) in Chapter 5. Truth is we are much more interested in a yet *another* transform called the Discrete Fourier Transform (DFT), but it is much easier to understand DFT if we start with the continuous-time case first. So although you will come across CTFT only in books and school, it is essential for the full understanding of this topic.

In Eq. (4.1) we give the expression for the Fourier series coefficients of a continuous-time signal from chapter 2.

$$\boxed{C_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jk\omega_0 t}dt} \tag{4.1}$$

To apply this to the aperiodic case, we let T_0 go to ∞. In Eq. (4.1) as the period gets longer, we are faced with division by infinity. Putting the period in form of frequency avoids this problem. Then we only have to worry about multiplication by zero. We write the period as a function of the frequency.

$$\frac{1}{T_0} = \frac{\omega_0}{2\pi} \tag{4.2}$$

If T_0 is allowed to go to infinity, then ω_0 is becoming tiny. In this case, we write frequency ω_0 as $\Delta\omega$ instead, to show that it is changing and getting smaller. Now we write the period in the limit as

$$\lim_{T_0\to\infty} \frac{1}{T_0} \approx \frac{\Delta\omega}{2\pi} \tag{4.3}$$

We rewrite Eq. (4.1) by substituting Eq. (4.3).

$$C_k = \frac{\Delta\omega}{2\pi}\int_{-T_0/2}^{T_0/2} x(t)e^{-jk\omega_0 t}dt \tag{4.4}$$

But now as T_0 goes to infinity, $\Delta\omega$ approaches zero, and the whole expression goes to zero. To get around this problem, we start with the time-domain Fourier series representation of $x(t)$, as given by

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\omega_0 t}} \tag{4.5}$$

Now substitute Eq. (4.4) into Eq. (4.5), the value of C_k becomes

$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{\left\{ \lim_{T_0\to\infty} \frac{\Delta\omega}{2\pi}\int_{-T_0/2}^{T_0/2} x(t)e^{-j\omega t}dt \right\}}_{C_k} e^{jk\omega_0 t} \tag{4.6}$$

In this expression, the summation can be replaced by an integral because we are now multiplying the coefficients (the middle part) with dt, sort of like computing an infinitesimal area. We change $\Delta\omega$ to $d\omega$, and $k\omega_0$ to ω, the continuous frequency. We also move the factor $1/2\pi$ outside. Now we rewrite Eq. (4.6) incorporating these ideas as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{\left\{\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right\}} e^{j\omega t}d\omega \tag{4.7}$$

We give the underlined part a special name, calling it the Fourier transform and refer to it by $X(\omega)$. Substituting this nomenclature in (4.7) for the underlined part, we write it in a new form. This expression is called the **Inverse Fourier transform** and is equivalent to the Fourier series representation or the synthesis equation.

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{X(\omega)} e^{j\omega t}d\omega} \tag{4.8}$$

The Continuous-Time Fourier Transform (CTFT) is defined as the underlined part in (4.8) and is equal to

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt} \tag{4.9}$$

In referring to the Fourier transform, the following terminology is often used.

If $x(t)$ is a time function, then its Fourier transform is written with a capital letter. Such as for time-domain signal, $y(t)$ the CTFT would be written as $Y(\omega)$. These two terms are called a **transform pair** and often written with a bidirectional arrow in between them such as here.

$$\begin{array}{lclclcl} y(t) & \leftrightarrow & Y(\omega) & \quad & y(t) & \xleftrightarrow{\mathfrak{F}} & Y(\omega) \\ c(t) & \leftrightarrow & C(\omega) & \quad & c(t) & \xleftrightarrow{\mathfrak{F}} & C(\omega) \end{array}$$

The symbol $\mathfrak{F}\{\cdot\}$ is also used to denote the Fourier transform. The symbol $\mathfrak{F}^{-1}\{\cdot\}$ is used to denote the inverse transform such that

$$\begin{aligned} Y(\omega) &= \mathfrak{F}\{y(t)\} \\ g(t) &= \mathfrak{F}^{-1}\{G(\omega)\} \end{aligned}$$

The CTFT is generally a complex function. We can plot the real and the imaginary parts of the transform, or we can compute and plot the magnitude, referred to as $|X(\omega)|$ and the phase, referred to as $\angle X(\omega)$. The magnitude is computed by taking the square root of the product $X(\omega)X^*(\omega)$ and phase by the arctan of the ratio of the imaginary and the real parts. We can also write the transform this way, separating out the magnitude and the phase spectrums.

$$X(\omega) = |X(\omega)|e^{j\angle X(\omega)}$$

Here

- Magnitude Spectrum: $|X(\omega)|$
- Phase Spectrum: $\angle X(\omega)$.

Comparing Fourier series coefficients and the Fourier transform

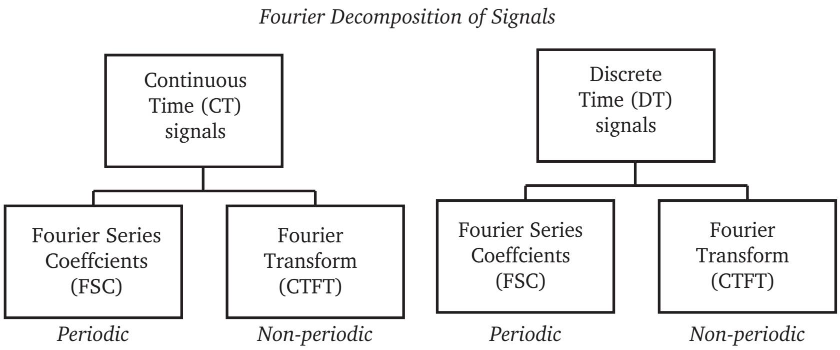

Figure 4.4: Fourier series and the Fourier transform

We can use the Fourier series analysis with both discrete and continuous-time signals as long as they are periodic. When a signal is aperiodic, the premium tool of analysis is the **Fourier transform**. Just as Fourier series can be applied to continuous and discrete signals, the Fourier transform can also be applied to continuous and discrete signals. The discrete version of the Fourier transform is called the DTFT and we will discuss it in the next chapter.

Let's compare the continuous-time Fourier transform (CTFT) with the Fourier series coefficient (FSC) equations. The FSC and the CTFT are given as:

$$\begin{aligned} \text{FSC: } C_k &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jk\omega_0 t}dt \\ \text{CTFT: } X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \end{aligned} \tag{4.10}$$

When we compare FSC with the CTFT expressions, we see that they are nearly the same except the period T_0 in the front of FSC is missing from the latter. Where did it go and does it have any significance? We started development of CTFT (see Fig. 4.2) by stretching the period and allowing it to go to infinity. We also equated $1/T_0$ to $d\omega/2\pi$ which was then associated with the time-domain formula or the inverse transform (notice, it is not included in the center part of Eq. (4.8), which became the Fourier transform.). So it moved to the inverse transform along with the 2π factor.

Notice now the difference between the time-domain signal representation as given by Fourier series and the Fourier transform.

$$\begin{aligned} \text{FSC: } x(t) &= \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \\ \text{CTFT: } x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \end{aligned} \tag{4.11}$$

In Fourier series representation, to determine the quantity of a particular harmonic, we multiplied the signal by that harmonic, integrated the product over one period and divided the result by the fundamental period T_0. This gave us the amplitude of that harmonic. In fact we did that for all harmonics, each divided by T_0. But here in Fourier transform, we do not divide by the period because we *don't know what it is*. We assumed that it is ∞, but we would not want to divide by that either. So we just ignore it and hence we are not determining the signal's true amplitude. We are computing a measure of the content but it is not the actual content. And since we are missing the same term from all coefficients, we say that, the Fourier transform determines relative amplitudes. But often that is good enough. All we are really interested in are the relative levels of harmonic signal powers. The true content of the harmonic signals in most cases is not important. Fourier spectrum gives us the *relative distribution* of power among the various harmonic frequencies in the signal. In practice, we often normalize the maximum power to 0 dB such that the relative levels are consistent among all frequency components.

CTFT of important aperiodic functions

Now we will take a look at some important aperiodic signals and their transforms, also called transform pairs. In the process, we will use the properties listed in Table 4.1. which can be used to compute the Fourier transform of many functions. The properties listed in Table 4.1 can be used to simplify computation of many transforms. We won't prove these properties but will refer to them as needed for the following important examples. These examples cover the fundamental functions that come up both in workplace DSP as well in textbooks, so they are worth understanding and memorizing. We will use the properties listed in Table 4.1

to compute the CTFTs in the subsequent examples in this and the following chapters. All following examples assume that the signal is aperiodic and is specified in continuous-time. The Fourier transform in these examples is referred to as CTFT.

Table 4.1: Important CTFT properties

Zero value	$X(0)=\int_{-\infty}^{\infty} x(t)dt$
Duality	If $x(t) \longleftrightarrow X(\omega)$, then $X(t) \longleftrightarrow x(\omega)$
Linearity	$ax(t)+by(t) \longleftrightarrow aX(\omega)+bY(\omega)$
Time Shift	$x(t-t_0) \longleftrightarrow e^{-j\omega t_0}X(\omega)$
Frequency Shift	$e^{j\omega_0 t}x(t) \longleftrightarrow X(\omega-\omega_0)$
Time Reversal	$x(-t) \longleftrightarrow X(-\omega)$
Time expansion or contraction	$x(at) \longleftrightarrow \frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right),\ a \neq 0.$
Derivative	$\frac{d}{dt}x(t) \longleftrightarrow j\omega X(\omega)$
Convolution in time	$x(t)*h(t) \longleftrightarrow X(\omega)H(\omega)$
Multiplication in Time	$x(t)y(t) \longleftrightarrow X(\omega)*Y(\omega)$
Power Theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$

CTFT of an impulse function

Example 4.1.

$$x(t)=\delta(t) \tag{4.12}$$

This is the most important function in signal processing. The delta function can be considered a continuous (Dirac delta function) or a discrete function (Kronecker delta function), but here we treat it as a continuous function.

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt \\
&= e^{-j\omega(t=0)} \\
&= 1
\end{aligned}$$

We use the CTFT expression in Eq. (4.9) and substitute delta function for function $x(t)$. In the third step, we used the sifting property of the delta function. The sifting property states that the integral of the product of a continuous-time signal with a delta function isolates the value of the signal at the location of the delta function per Eq. (4.13).

$$\int_{-\infty}^{\infty} \delta(t-a)x(t)dt = x(a). \tag{4.13}$$

If $a = 0$, then the isolated value of the complex exponential is 1.0, at the origin. The integrand becomes a constant, so it is no longer a function of frequency. Hence CTFT is constant for all frequencies. We get a flat line for the spectrum of the delta function.

The delta function was defined by Dirac as a summation of an infinite number of exponentials.

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}dt \tag{4.14}$$

The general version of (4.14) with a shift is given as

$$\boxed{\delta(t-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j(\omega-a)t}dt} \tag{4.15}$$

In its transform, we see a spectrum that encompasses whole of the frequency space to infinity, hence a flat line from $-\infty$ to $+\infty$. In fact when in Chapter 1, Fig. (1.9), we added a whole bunch of sinusoids, this is just what we were trying to get at. This is a very important property to know and understand. It encompasses much depth and if you understand it, the whole of signal processing becomes easier.

Now if the CTFT or $X(\omega) = 1$ then what is the inverse of this CTFT? We want to find the time-domain function that produced this function in frequency domain. It ought to be a

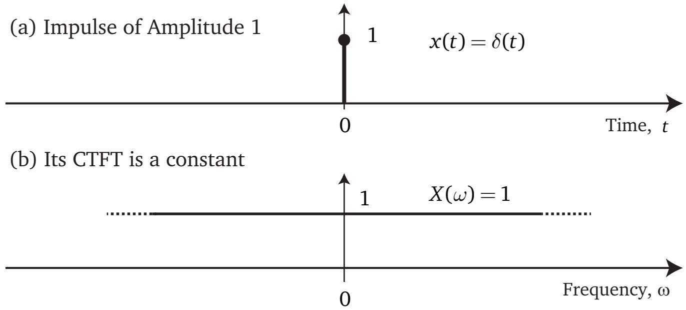

Figure 4.5: CTFT of a delta function located at time 0 is a constant.

delta function but let's see if we get that. Using the inverse CTFT Eq. (4.8), we write

$$\begin{aligned} x(t) &= \mathfrak{F}^{-1}\{1\} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{j\omega t}\, dt \\ &= \delta(t) \end{aligned}$$

Notice that we already showed that the summation of complex exponential leads to a delta function. Substituting in the second step the definition of the delta function from Eq. (4.14), we get the function back. A perfect round trip. The CTFT of a delta function is 1 in frequency-domain. The inverse CTFT of 1 in frequency-domain is the delta function in time-domain.

$$\delta(t) \xrightarrow{\text{CTFT}} 1 \xrightarrow{\text{Inverse CTFT}} \delta(t)$$

CTFT of a constant

Example 4.2. What is the Fourier transform of the time-domain signal, $x(t) = 1$.

This case is different from Example 4.1. Here the time-domain signal is a constant and not a delta function. Using (4.9), we write the CTFT as

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} 1 \cdot e^{-j\omega t}\, dt \end{aligned}$$

Using Eq. (4.14) for the expression for the delta function, we get the CTFT of the constant 1 as

$$X(\omega) = 2\pi\delta(\omega)$$

It can be a little confusing as to why there is this 2π factor, but it is coming from the definition of the delta function, (4.14).

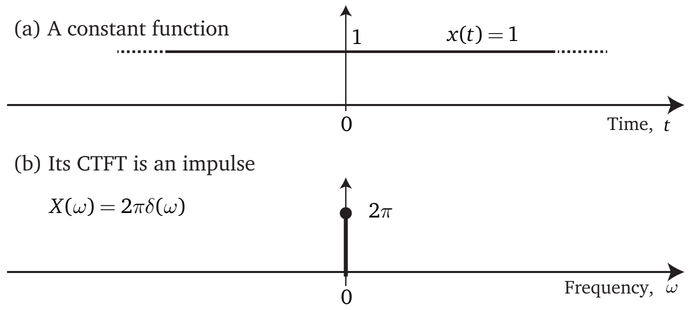

Figure 4.6: CTFT of a constant function which shows reciprocal relationship with example 4.1.

If the time-domain signal is a constant, then its Fourier transform is the delta function and if we were to do the inverse transform of $2\pi\delta(\omega)$ we would get back $x(t) = 1$. We can write this pair as

$$1 \longleftrightarrow 2\pi\delta(\omega)$$

Note in example 4.1, we had this pair $\delta(t) \longleftrightarrow 1$. Which is confusingly similar but is not the same thing.

CTFT of a sinusoid

Example 4.3. Since a sinusoid is a periodic function, we will select only one period of it to make it *aperiodic*. Here we have just a piece of a sinusoid. We make no assumption about what happens outside the selected time frame. The cosine wave shown in Fig. 4.7 has a frequency of 3 Hz, hence you see one period of time which is 0.33 s.

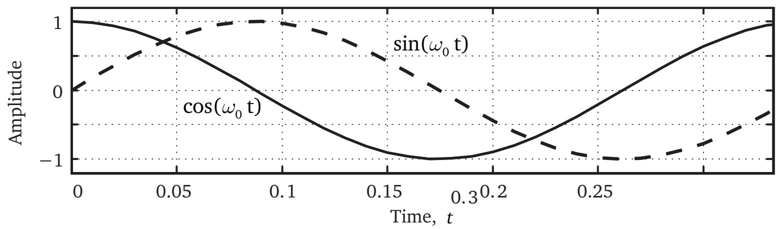

Figure 4.7: By limiting the duration of a sinusoid, we effectively make it an aperiodic signal.

We can compute the CTFT of this little piece of cosine as

$$\begin{aligned} X(\omega) &= \mathfrak{F}\{\cos(\omega_0 t)\} \\ &= \int_{-\infty}^{\infty} \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{j\omega t} dt \\ &= \int_{-\infty}^{\infty} \frac{1}{2} e^{j(\omega_0+\omega)t} dt + \int_{-\infty}^{\infty} \frac{1}{2} e^{j(\omega_0-\omega)t} dt \end{aligned}$$

This is the summation of two integrals. When we compare these to the Fourier transform of the delta function we see that they are similar. Applying the time-shift property from Table 4.1, we can show that the integrals are the Fourier transforms of shifted delta functions, from their definition in Eq. (4.15). The amount of shift in time is equal to the frequency of the cosine. From Eq. (4.14) we write the CTFT of this cosine piece as

$$X(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0) \tag{4.16}$$

The only difference we see between the CTFT of a cosine wave and FSC we see in Example 2.1 is the scaling. In the case of FSC, we get two delta functions of amplitude 1/2 for each or a total of 1. The amplitude of the CTFT is however π , which 2π times the amplitude of the FSC.

By similarity, the Fourier transform of a sine is given by

$$X(\omega) = \mathfrak{F}\{\sin(\omega_0 t)\} = j\pi\delta(\omega + \omega_0) - j\pi\delta(\omega - \omega_0) \tag{4.17}$$

Note the presence of j in front just means that this transform is in the Imaginary plane.

CTFT of a complex exponential

Example 4.4. Now we calculate the CTFT of the very important function, the complex exponential.

$$x(t) = e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

A CE is really two functions, one a cosine of frequency ω_0 and the other a sine of the same frequency, both orthogonal to each other.

We have already calculated the CTFT of a sine and a cosine in (4.15) given by

$$\begin{aligned} \mathfrak{F}\{\cos(\omega_0 t)\} &= \pi\delta(\omega - \omega_0\} + \pi\delta(\omega + \omega_0) \\ \mathfrak{F}\{\sin(\omega_0 t)\} &= -j\big(\pi\delta(\omega - \omega_0) - \pi\delta(\omega + \omega_0)\big) \end{aligned}$$

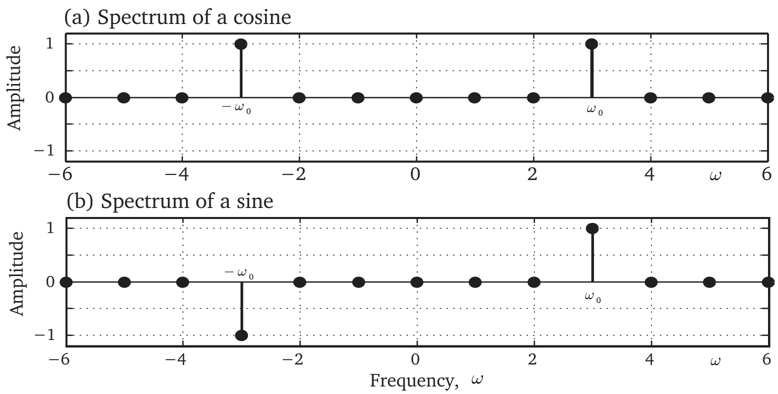

Figure 4.8: CTFT of the cosine and the sine function.
Had we plotted the magnitude, both plots would look identical.

By the linearity principle, we write the Fourier transform of the CE keeping the sine and cosine separate.

$$\begin{aligned}
\mathfrak{F}\{e^{j\omega_0 t}\} &= \mathfrak{F}\{\cos(\omega_0 t)\} + j\mathfrak{F}\{\sin(\omega_0 t)\} \\
&= \underbrace{\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)}_{\text{cos}} + \underbrace{(-j)\big(\pi\delta(\omega-\omega_0) - \pi\delta(\omega+\omega_0)\big)}_{\text{sin}} \\
&= \underbrace{\pi\delta(\omega-\omega_0) - j^2\pi\delta(\omega-\omega_0)}_{\text{add}} + \underbrace{\pi\delta(\omega+\omega_0) + j^2\pi\delta(\omega+\omega_0)}_{\text{cancel}} \\
&= 2\pi\delta(\omega-\omega_0)
\end{aligned} \tag{4.18}$$

In DSP we call the real and imaginary axes the I and Q channels. These come from the terms **I**n-phase and **Q**uadrature. A sine and cosine wave can exist together on a line and not interfere because they are orthogonal to each other. This is because they are 90° degrees out of phase, or said to be in quadrature. In baseband and in hardware it is much faster to take a signal, decompose it in these two orthogonal components and then do separate signal processing on each at half the rate, leading to speed improvement. In the receiver there is a local oscillator that matches the cosine portion of the signal. Therefore the cosine is called In-phase and the sine wave is in Quadrature.

In Cartesian sense, I and Q are on the x and y axes. When a number is purely on one of these axes, it has no component in the other. Any number in between can be projected on both the x and y axis. The projections are the x and y components. Now just think of x as the I axis and y as the Q axis. And instead of a scalar, think of a signal. Now the projections are the "amount" of cosine on the I axis and "amount" of sine on the Q axis. I axis represents

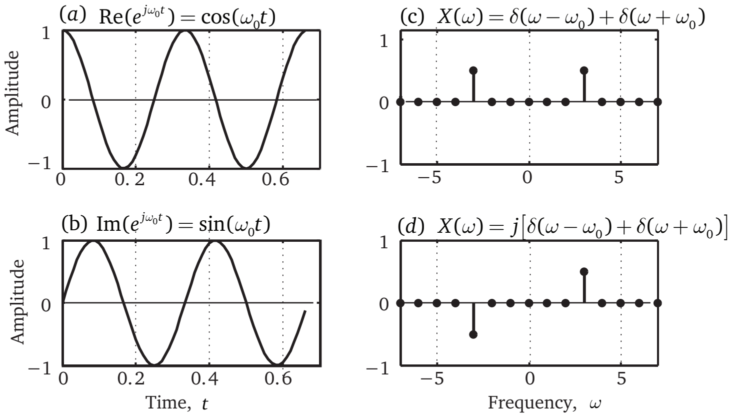

Figure 4.9: The CTFT of a complex exponential.
The Real part is a cosine, hence the spectrum looks like Fig. 4.8(a) and the Imaginary part is a sine, and hence this plot is exactly the same as Fig. 4.8(b).

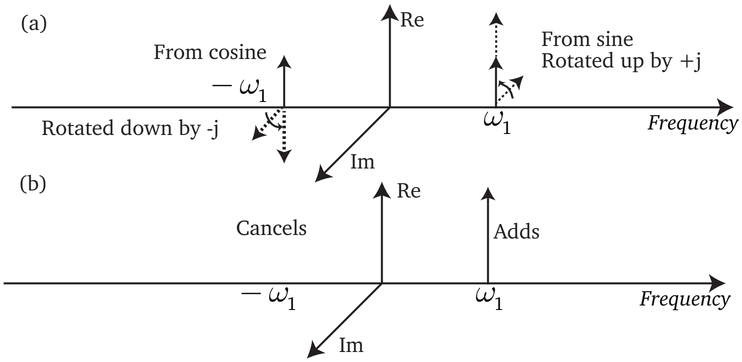

Figure 4.10: The asymmetrical spectrum of a complex exponential.

the cosine projection and Q axis the sine projection of a signal. When multiplying by j, the phase changes by 90°. This is same as moving from one axis to the other axis. Unfortunately to add to the confusion, the I in this terminology corresponds to the real axis, and Q to the imaginary axis!

Now consider an Inverse-CTFT (also referred to as taking the iCTFT) that consists of a single impulse located at frequency ω_1 written as

$$X(\omega) = \delta(\omega - \omega_1)$$

We want to know what time-domain function produced this spectrum. We take the iCTFT.

$$\begin{aligned} x(t) &= \mathfrak{F}\{\delta(\omega-\omega_1)\} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_1)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}e^{j\omega_1 t} \end{aligned}$$

The result is a complex exponential of frequency ω_1 in time-domain. Because this is a complex signal, it has a non-symmetrical frequency response which consists of just one impulse located at the CE's frequency. In Fig. 4.10, we see why it is one sided. The reason is that a cancellation occurs on the negative side and an addition on the positive side of the frequency axis. The component from the sine rotates up to add to cosine part on the positive and rotates down on the negative side to cancel the cosine portion. This of course is coming from the Euler's equation.

Here we write the two important CTFT pairs. The CTFT of a CE is one-sided, an impulse at its frequency. (The CTFT of a complex function comes out to be *real*.)

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega-\omega_0) \qquad \text{and} \qquad e^{-j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega+\omega_0).$$

Time-shifting a function

What is the CTFT of a delta function shifted by time t_0? We can construct many signals as summation of time-shifted delta functions. This case is very important to further understanding of discrete signals.

We can determine the response of a delayed signal by noting the time shift property in Table 4.1. The property says that if a function is delayed by a time period of t_0, then in frequency domain, the original response of the un-delayed signal is multiplied by a CE of frequency $e^{j\omega t_0}$. This is given by the product of $X(\omega)$ and $e^{j\omega t_0}$, where $X(\omega)$ is the Fourier transform of the un-delayed signal. Look carefully at this signal, $e^{j\omega t_0}$. Note that time is constant and hence this is a frequency-domain signal, with frequency as the variable.

We write the shifted signal as $x(t) = \delta(t-t_0)$. Calculate the Fourier transform of this function from Eq. (4.9) as

$$\begin{aligned}X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} \delta(t-t_0)e^{-j\omega t}dt \\ &= e^{-j\omega t_0}\end{aligned}$$

What we get in frequency domain for this delayed delta function is a CE. This CE has the form $e^{-j\omega t_0}$ and might be confusing. That's because we are not used to seeing exponentials in frequency domain. Now intuitively speaking, if you have a signal and you move it from one "place" to another, does anything change about the signal? Similarly, delaying a signal does not change its amplitude (the main parameter by which we characterize signals.) Its frequency also does not change, but what does change is its phase. If a sine wave is running and we arrive to look at it at time t_0 after it has started, we are going to see an instantaneous phase at that time which will be different depending on when we arrive on the scene. That's all a time shift does.

We can show this by computing the magnitude and the phase of the spectrum of a delayed signal. Here we write the magnitude of the delayed signal as the magnitude of the product of the original spectrum and the CE $e^{j\omega t_0}$.

$$\begin{aligned}\text{Mag}\big\{\mathfrak{F}\{x(t-t_0)\}\big\} &= \left|e^{-j\omega t_0}X(\omega)\right| \\ &= \left|e^{-j\omega t_0}\right|\left|X(\omega)\right|\end{aligned}$$

We compute the magnitude of the CE first. Keep in mind that this is not a time-domain signal. The variable is frequency, and not time. The CE is given in the Euler form as

$$e^{j\omega t_0} = \cos(\omega t_0) + j\sin(\omega t_0)$$

We compute the magnitude of this signal by

$$\begin{aligned}\left|e^{-j\omega t_0}\right| &= \left|\cos(\omega t_0) + j\sin(\omega t_0)\right| \\ &= \cos^2(\omega t_0) + \sin^2(\omega t_0) \\ &= 1\end{aligned}$$

Being a sinusoid, even if a complex one in frequency-domain, its magnitude is still 1.0 because it is after all composed of two sinusoids. Then we compute the magnitude of the shifted function, which is the product of the un-delayed signal magnitude and the magnitude of the CE.

$$\begin{aligned}
\text{Mag}\big\{\mathfrak{F}\{x(t-t_0)\}\big\} &= \left|e^{-j\omega t_0}X(\omega)\right| \\
&= 1 \cdot \left|X(\omega)\right| \\
&= \sqrt{X(\omega)X^*(\omega)} \\
&= \left|X(\omega)\right|
\end{aligned}$$

The magnitude of the delayed signal is same as that of the un-delayed signal. Delay does not change the magnitude. From the same equation, we see that the magnitude is a function of the same frequency variable and has not been modified by the process. So what did change by shifting the signal in time? Now we look at the phase. The phase delay was

$$\phi_{\text{undelayed}} = \tan^{-1}\frac{\text{Im}\{X(\omega)\}}{\text{Re}\{X(\omega)\}}$$

The phase response of the delayed signal is given by

$$\begin{aligned}
\phi_{\text{delayed}} &= \tan^{-1}\frac{\text{Im}\{e^{j\omega t_0}X(\omega)\}}{\text{Re}\{e^{j\omega t_0}X(\omega)\}} \\
&= \tan^{-1}\underbrace{\frac{\sin(\omega t_0)}{\cos(\omega t_0)}}_{\text{function of } t_0} \cdot \frac{\text{Im}\{X(\omega)\}}{\text{Re}\{X(\omega)\}}
\end{aligned}$$

Since this coefficient is a function of the delay, the phase has indeed changed from the un-delayed case. The conclusion we draw is that delaying a signal changes its phase response, or equivalently multiplication by a CE in the frequency domain changes the phase of a signal. This property is used in simulation to add phase shifts to a signal.

In general, if we shift a signal by time t_0, the Fourier transform of the signal can be calculated by the time shift property as

$$x(t-t_0) \longleftrightarrow e^{j\omega t_0}X(\omega) \tag{4.19}$$

In Fig. 4.11 we show the effect of time-delay. In (a), we have a signal with an arbitrary spectrum centered at frequency of 2 Hz. We don't actually show this signal, only its Fourier

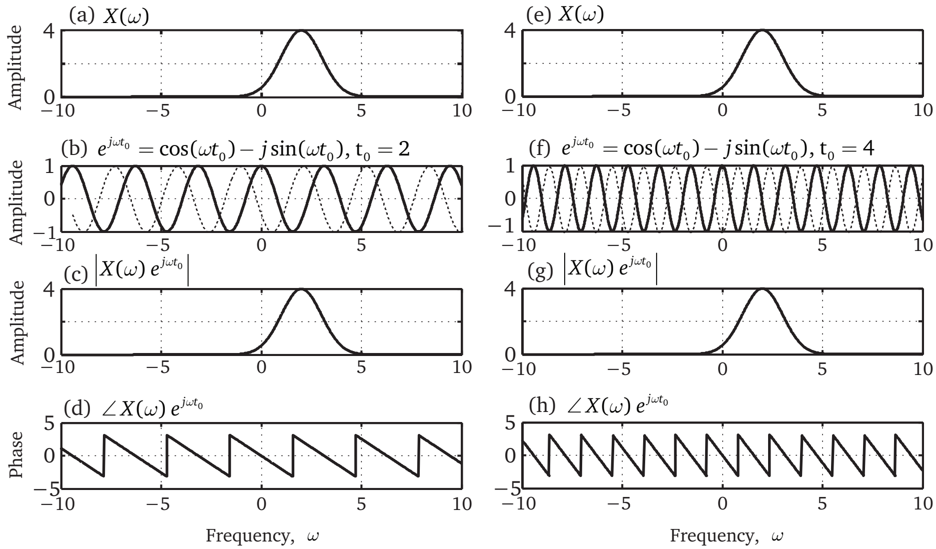

Figure 4.11: Signal delay causes only the phase response to change. In (a) we see an arbitrary signal delayed by 2 seconds and in (b), it has been delayed by 4 seconds in (e). Both cases have the same magnitude but the phase is different.

transform, with x-axis being frequency. You only need to note its shape and center location on the frequency axis. Now we delay this signal by 2 seconds (we don't know what the signal is, but that does not matter.) and want to see what happened to the spectrum.

In (b) we draw the CE $e^{j\omega t_0}$ with $t_0 = 2$ (both sine and cosine are shown). In (c), we see the effect of multiplying this CE by the spectrum in (a). The magnitude is unchanged. But when we look at (d) we see the phase. Since we do not know what the previous phase was, no statement can be made about it yet. Now examine the second column. In this case the signal is delayed by 4 seconds. Once again in (g) we see no change in magnitude but we see that phase in (h) has indeed changed from pervious case in (d).

Duality with frequency shift

If a signal is shifted in time, the response changes for phase but not for frequency. Now what if we shift the spectrum by frequency such as $X(\omega)$ vs. $X(\omega - \omega_0)$, i.e. the response is to be shifted by a constant frequency shift of ω_0. We can do this by using the frequency shift property. The CTFT of the frequency-shifted signal changes as

$$X(\omega - \omega_0) \longleftrightarrow e^{j\omega_0 t} x(t) \tag{4.20}$$

If we multiply a time-domain signal by a CE of a desired frequency, the result is a shifted frequency response by the new frequency.

$$\begin{aligned}\mathfrak{F}\{e^{j\omega_0 t}x(t)\} &= \int_{-\infty}^{\infty} e^{j\omega_0 t}x(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt \\ &= X(\omega-\omega_0).\end{aligned} \tag{4.21}$$

Neither the frequency nor the time shifts change the magnitude of the spectrum. The only thing that changes is the phase. The frequency shift property is also called the modulation property. We think of modulation as multiplying a signal by a carrier and in-fact if you look at Eq. (4.20), that is exactly what we are doing. The CE can be thought of as the carrier signal, a complex sinusoid of a single carrier frequency. A time-domain signal multiplied by a such a CE, $e^{j\omega_0 t}$ results in the signal transferring (or upconverted) to the carrier frequency without change in its amplitude.

Convolution property

The most important result from Fourier transform is the convolution property. In fact Fourier transform is often used to perform convolution in hardware instead of doing convolution in time-domain. The property is given by

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \tag{4.22}$$

In time-domain, convolution is a resource-heavy computation. Calculating integrals is more costly in terms of time than multiplication. But convolution can be done using the Fourier transform convolution property. The convolution property states:

$$x(t) * h(t) \longleftrightarrow X(\omega)H(\omega) \tag{4.23}$$

This says that the convolution of two signals can be computed by multiplying their individual Fourier transforms and then taking the inverse transform of the product. In many cases this is simpler to do. We can prove this as follows. We will write the time-domain expression for the convolution and then take its Fourier transform. Yes, it does look messy and requires fancy calculus.

$$\begin{aligned}\mathfrak{F}\{x(t) * h(t)\} &= \mathfrak{F}\left\{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau e^{-j\omega t}dt\end{aligned}$$

Now we interchange the order of integration to get this from

$$\mathfrak{F}\{x(t)*h(t)\} = \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right)d\tau$$

We make a variable change by setting $u = t-\tau$, hence we get

$$\mathfrak{F}\{x(t)*h(t)\} = \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} h(u)e^{-j\omega(u+\tau)}du\right)d\tau$$

This can be written as

$$\mathfrak{F}\{x(t)*h(t)\} = \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} h(u)e^{-j\omega u}e^{-j\omega\tau}du\right)d\tau$$

Now we move the $e^{-j\omega\tau}$ term out of the inner integral because, it is not function of u, to get the desired result and complete the convolution property proof.

$$\begin{aligned}\mathfrak{F}\{x(t)*h(t)\} &= \left(\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau\right)\left(\int_{-\infty}^{\infty} h(u)e^{-j\omega u}du\right)\\ &= X(\omega)H(\omega)\end{aligned}$$

The duality property of Fourier transform then implies that if we multiply two signals in time-domain, then the Fourier transform of their product would be equal to convolution of the two transforms.

$$x(t)h(t) \longleftrightarrow \frac{1}{2\pi}X(\omega)*H(\omega) \tag{4.24}$$

This is an efficient way to compute convolution. Convolution can be hard to visualize. The one way to think of it is as smearing or a smoothing process. The convolution process produces the smoothed version of one of the signals as we can see in Fig. 4.12.

CTFT of a Gaussian function

Example 4.5. Now we examine the CTFT of a really unique and useful function, the Gaussian. The zero-mean Gaussian function is given by

$$x(t) = \frac{1}{\sigma\sqrt{2\pi}}e^{-t^2/(2\sigma^2)}$$

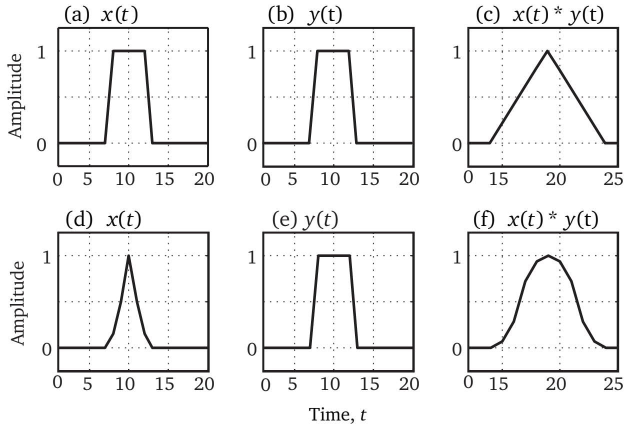

Figure 4.12: (a) The convolution of signals $x(t)$ and $y(t)$ in (c) is done using Fourier transform. In each case the result is smoother than either of the original signals. Hence convolution can be thought of as a filter.

where σ^2 is the signal variance and σ the standard deviation of the signal. The CTFT of this function is very similar to the function itself.

$$X(\omega) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/(2\sigma^2)} e^{j\omega t} dt$$
$$= \frac{1}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2\sigma^2} e^{-j\omega t} dt$$

This is a difficult integral to solve but fortunately smart people have already done it for us. The result is

$$X(\omega) = \frac{1}{2\sigma\sqrt{2\pi}} e^{-\frac{\sigma^2\omega^2}{2}}$$

Since σ is a constant, the shape of this curve is a function of the square of the frequency, same as it is in time-domain where it is square of time. Hence it is often said that the CTFT of the Gaussian function is same as itself, but what they really mean is that the shape is the same. This property of the Gaussian function is very important in nearly all fields.

CTFT of a square pulse

Example 4.6. Now we examine the CTFT of a square pulse of amplitude 1, with a period of τ, centered at time zero. This case is different from the ones in Chapter 2 and Chapter 3 in that here we have just a single solitary pulse. This is not a case of repeating square pulses since in this section we are considering only aperiodic signals.

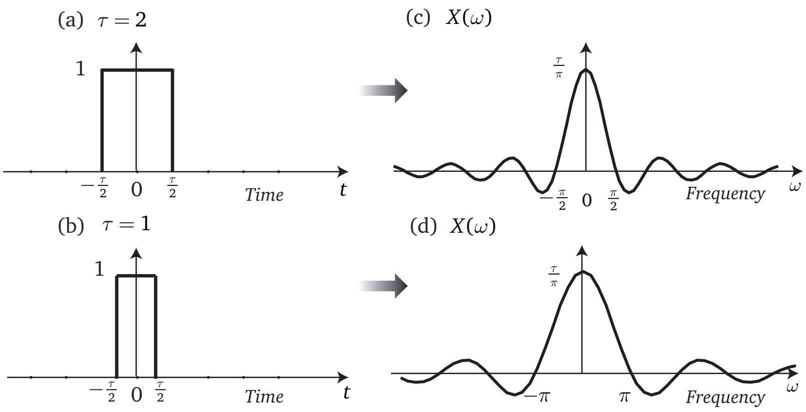

Figure 4.13: Spectrum along the frequency line. A square pulse has a sinc-shaped spectrum. (a) its time-domain shape, (b) its CTFT.

We write the CTFT as given by Eq. (4.9). The function has an amplitude of 1.0 for the duration of the pulse. Hence integration takes place over half of the period.

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-\tau/2}^{\tau/2} 1 \cdot e^{-j\omega t}dt \\
&= -\frac{e^{-j\omega t}}{j\omega}\bigg|_{-\tau/2}^{\tau/2} \\
&= -\frac{1}{j\omega}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right) \\
&= \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right)
\end{aligned}
$$

This can be simplified to

$$X(\omega) = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right) \tag{4.25}$$

We see the spectrum plotted in Fig. 4.13 for $t = 1s$ and $t = 2s$. Note that as the pulse gets longer (or wider), its spectrum gets narrower. Since the sinc function is zero for all values that are integer multiple of 2π, the zero crossings occur whenever $\omega\tau = k\pi$, where k is an even integer larger than 2. For $t = 2$ the zeros would occur at radial frequency equal to $\pi, 2\pi, \ldots$. If the pulse were to become infinitely wide, the CTFT would become an impulse function. If it were infinitely narrow as in Example 4.1, the frequency spectrum would be flat.

Now assume that instead of the time-domain square pulse shown in Fig. 4.13, we are given a frequency response that looks like a square pulse. The spectrum is flat from $-W$ to $+W$ Hz. This can be imagined as the frequency response of an ideal filter. Notice, that in the time-pulse case, we defined the half-width of the pulse as $\tau/2$, but here we define the half bandwidth by W and not by $W/2$. The reason is that in time-domain, when a pulse is moved, its period is still τ. But bandwidth is designated as a positive quantity only. There is no such thing as a negative bandwidth. In this case, the bandwidth of the signal (because it is centered at 0 is said to be W Hz and not $2W$ Hz. However if this signal were moved to a higher frequency such that the whole signal was in the positive frequency range, it would be said to have a bandwidth of $2W$ Hz. This crazy definition gives rise to the concepts of low-pass and band-pass bandwidths. Lowpass is centered at the origin so it has half the bandwidth of bandpass.

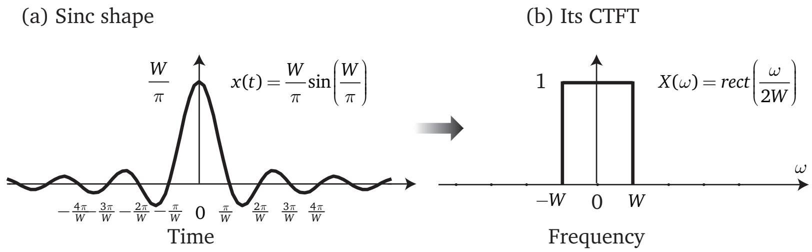

Figure 4.14: Time-domain signal corresponding to the rectangular frequency response.

What time-domain signal produces a rectangular frequency response shown in Fig. 4.14? The frequency response is limited to a certain bandwidth.

$$X(\omega) = \begin{cases} 1 & |\omega| \le W \\ 0 & |\omega| > W \end{cases}$$

We compute the time-domain signal by the inverse CTFT equation.

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\frac{e^{j\omega t}}{jt}\bigg|_{-W}^{W} \end{aligned}$$

Which can be simplified to

$$x(t) = \frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right)$$

Again we get a sinc function, but now in time-domain. This is the duality principle at work. This is a very interesting case and of fundamental importance in communications. A time-domain sinc shape has a very sharply defined response in the frequency-domain. But a sinc function looks strange for a time-domain signal because it is of infinite length. But because it is "well-behaved", which means it crosses zeros at predicable points, we can use it as a signal shape, at least in theory. In practice it is impossible to build a signal shape of infinite time duration. An alternate shape with similar properties is the **raised cosine**, most commonly used signal shape in communications.

The frequency spectrum shown in Fig. 4.14(b) is a very desirable frequency response. We want the frequency response to be tightly constrained. The way to get this type of spectrum is to have a time-domain signal that is a sinc function. This is the dual of the first case, where a square pulse produces a sinc frequency response.

In Fig. 4.15 we give some important Fourier transforms of non-periodic signals.

Fourier transform of periodic signals

Fourier transform came about so that the Fourier series could be made rigorously applicable to aperiodic signals. The signals we examined in this chapter so far were all *aperiodic*, even the cosine wave, which we limited to one period. Can we use the CTFT for a *periodic* signal? Our intuition says that this should be the same as the Fourier series. Let's see if that is the case.

Take a periodic signal $x(t)$ with fundamental frequency of $\omega_0 = 2\pi/T_0$ and write its FS representation.

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\omega_0 t}$$

Taking the CTFT of both sides of this equation, we get

$$X(\omega) = \mathfrak{F}\{x(t)\} = \mathfrak{F}\left\{\sum_{k=-\infty}^{\infty} C_k e^{j\omega_0 t}\right\}$$

We can move the coefficients out of the CTFT because they are not function of frequency. They are just numbers.

$$X(\omega) = \sum_{k=-\infty}^{\infty} C_k \mathfrak{F}\{\underline{e^{j\omega_0 t}}\}$$

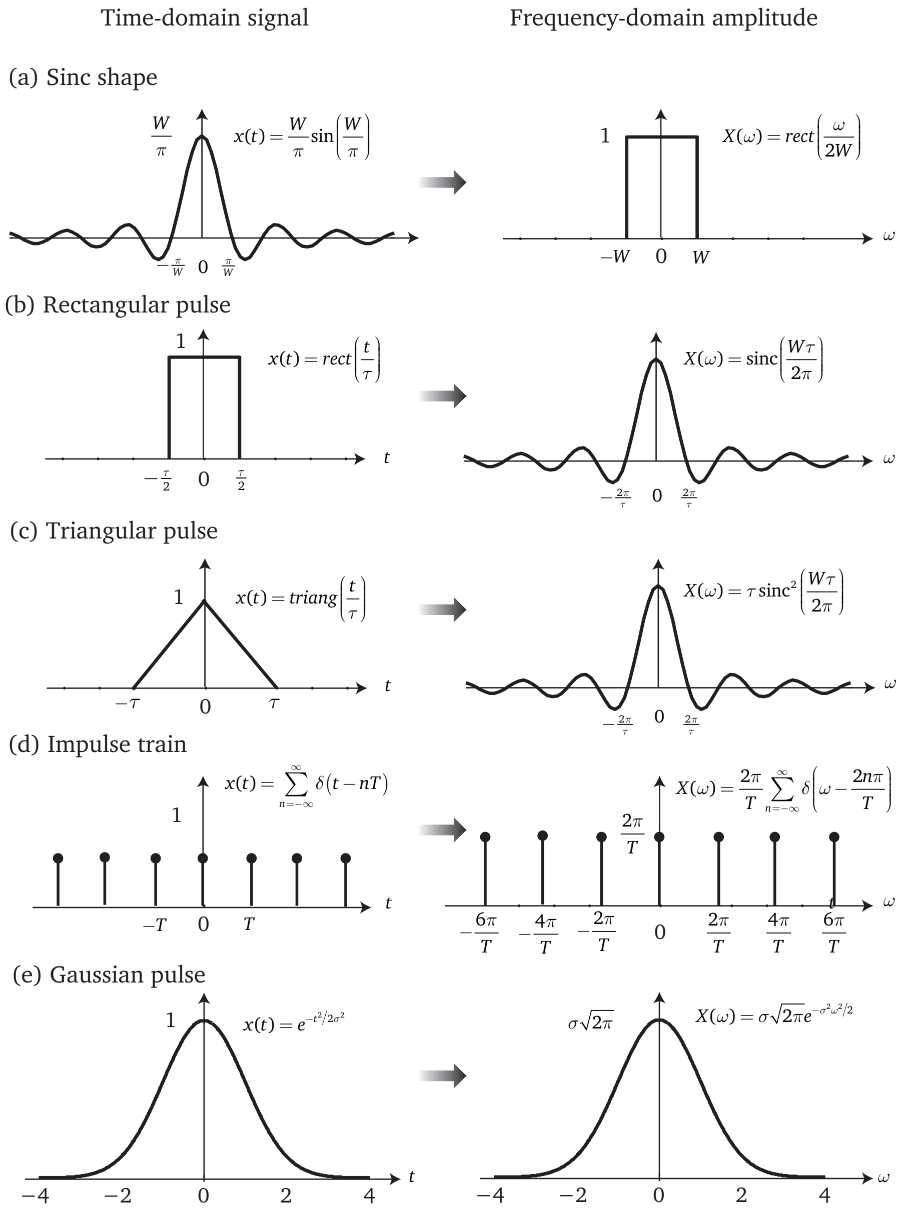

Figure 4.15: Response to a Sinc shaped time domain signal.

The Fourier transform of the complex exponential $e^{j\omega_0 t}$ is a delta function located at the frequency ω_0 as we saw in Example 4.4. Making the substitution, we get

$$\boxed{X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega - k\omega_0)} \tag{4.26}$$

What does this equation say? It says that the CTFT of a periodic signal is a sampled version of the Fourier series coefficients. But the Fourier series coefficients are already discrete! So the only thing the Fourier transform does is change the scale. The magnitude of the CTFT of a periodic signal is 2π times bigger than those computed with FS that you see in front of Eq. (4.26).

Important observation: The CTFT of an aperiodic signal is aperiodic but continuous whereas the CTFT of a periodic signal is also aperiodic but discrete.

CTFT of a periodic square pulse train

Example 4.7. Now we examine the CTFT of the periodic square pulse. For the Fourier transform of this periodic signal, we will use Eq. (4.26)

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega - k\omega_0)$$

The FSC of a periodic pulse train with duty cycle = 1/2 are computed in chapter 2 and given as

$$C_k = \frac{1}{2}\text{sinc}(k\pi/2)$$

We plot these Fourier series coefficients in Fig. 4.16.

To compute CTFT, we set $\omega_0 = 1$ and now we write the CTFT expression as

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} C_k \delta\big(\omega - k(\omega_0 = 1)\big)$$

The result is the sampled version of the Fourier series coefficients scaled by 2π (See Example 2.10) which are of course themselves discrete.

What if the square pulse was not centered at 0 but shifted some amount. We can compute the CTFT of this periodic function by applying the time shift property to the CTFT of the un-shifted square wave.

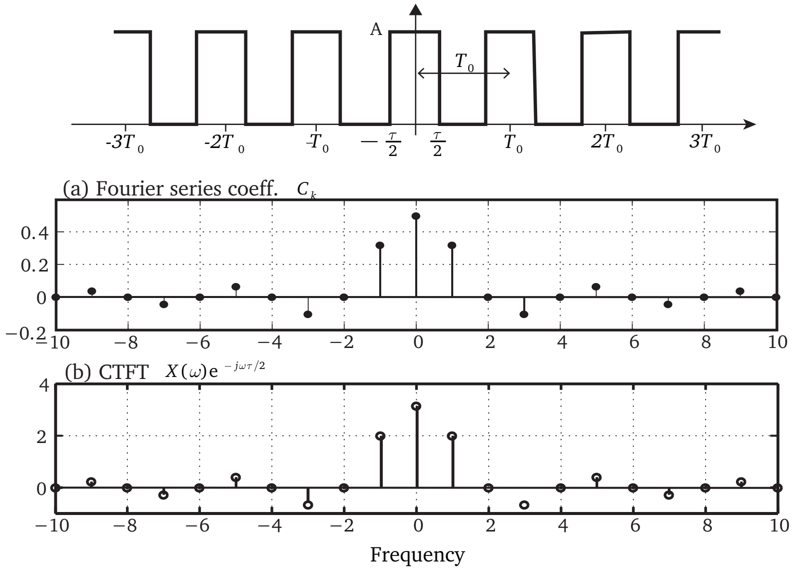

Figure 4.16: The periodic square wave with duty cycle of 0.5. *(b) Its FSC and (c) its CTFT. Only the scale is different.*

This periodic function is same as Fig. 4.16 but is time-shifted. We can write it as

$$y(t) = x(t - \tau/2)$$

By the time-shift property, we can write the CTFT of this signal by multiplying the CTFT of the un-shifted case by $e^{j\omega\tau/2}$. Hence

$$Y(\omega) = X(\omega)e^{j\omega\tau/2}$$

Which is

$$Y(\omega) = \left(2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega - k)\right) e^{j\omega\tau/2}$$

This time-shift has no effect on the shape of the response at all, just as we would expect. Only the phase gets effected by the time shift.

The main reason we have a Fourier transform vs. the Fourier series representation and its coefficients, is that the Fourier transform can be used for aperiodic signals. However since in developing the Fourier transform we have let go of the concept of a period, the results are useful in a relative sense only. The same happens when we try to manipulate the Fourier transform for use with periodic signals. Fourier transform hence becomes not a tool for accurately measuring the signal amplitudes as we might do in a scale, it becomes a qualitative tool.

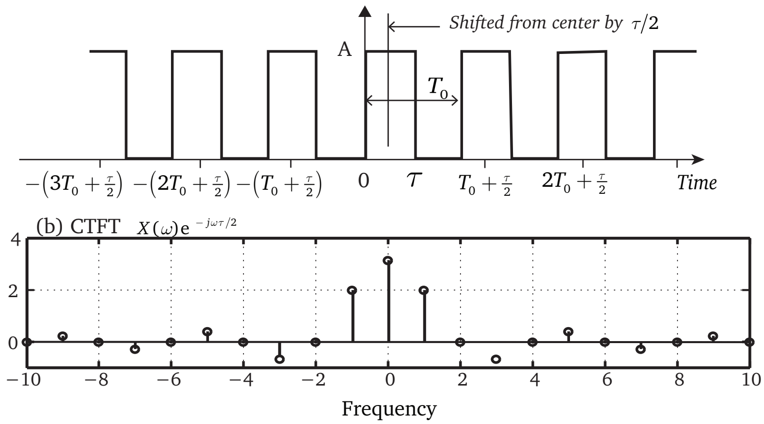

Figure 4.17: A time-shifted square pulse train.

Summary of Chapter 4

In this chapter we looked at aperiodic signals and their frequency representations. The FS concept is extended so that same analysis can be applied to aperiodic signals. In a manner similar to computing the coefficients, we call the process of computing the "coefficients" of aperiodic signal the Fourier transform. The spectrum of continuous signals using the Fourier transform is continuous, where the Fourier transform of a periodic signal is discrete.

Terms used in this chapter:

- **Fourier Transform, FT**
- **Continuous-time Fourier Transform, CTFT**
- **Discrete-time Fourier Transform, DTFT**
- **Transform pair** - The signal in one domain and its Fourier transform in the other domain are called a Fourier transform pair.

1. It is not mathematically valid to compute Fourier series coefficients of an aperiodic signal.
2. Fourier transform is developed by assuming that the period becomes infinite.
3. Where the spectrum of a periodic signal computed with the Fourier series is called the coefficients, the spectrum for an aperiodic signal is called the Fourier transform.
4. For continuous signals, it is called the CTFT and DTFT for discrete signals.
5. The CTFSC are discrete where the CTFT of an aperiodic signal is continuous in the frequency domain.
6. CTFT is aperiodic just as is the CTFSC.

7. The continuous-time Fourier transform is computed by

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

8. A function and its Fourier transform are called a Fourier transform pair.

$$x(t) \longleftrightarrow X(\omega)$$

9. The CTFT of a periodic signal is given by the expression

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} C_k\delta(\omega - k\omega_0)$$

10. The two representations, in time and frequency domain, CTFT and iCTFT are called a transform pair.
11. The CTFT is nearly identical to CTFSC. The values of CTFT are a factor 2π greater than the Fourier series coefficients calculated for the same signal and are sampled at the frequency of the signal.

Questions

1. What is the conceptual difference between the Fourier series and the Fourier transform?
2. Why is the CTFT continuous? Why are the CTFSC coefficients discrete?
3. What is the CTFT of these impulse functions:
   $\delta(t-1), \delta(t-2), \delta(t-T)$.
4. Give the expression for the CTFT of a cosine and a sine. What is the main difference between the two?
5. Given a sinusoid of frequency 5 Hz. What does its CTFT look like?
6. What is the difference between the Fourier transform (the magnitude) of a sine and a cosine of equal amplitudes?
7. What is CTFT (amplitude) of these sinusoids: $\sin(-800\pi t)$, $-\cos(250\pi t)$, $0.25\sin(25\pi t)$.
8. What is magnitude spectrum of these sinusoids?
9. What is the value of $\sin(500\pi t)\delta(t)$?
10. If the FT of a signal is being multiplied by this CE: $e^{(-j6\pi f)}$, what is the resultant effect in time domain?
11. We multiply a signal in time domain by this CE: $e^{(-j12\pi t)}$, what is the effect in frequency domain on the FT of the signal?

12. The summation of complex exponentials represent what other function?
13. What is the value of $\cos(6\pi t) * \delta(t-4)$?
14. What is the CTFT of the constant π?
15. A sinc function crosses first zero at π/B. What is its time domain equation? What does the spectrum look like and what is its bandwidth?
16. A sinc function crosses first zero at t = 1, give its time domain equation? What is its time domain equation? What does the spectrum look like and what is its bandwidth?
17. What is the CTFT of $\sin(5\pi t) * \delta(t-5)$?
18. A signal of frequency 4 Hz is delayed by 10 seconds. By what CE do you multiply the un-shifted CTFT to get the CTFT of the shifted signal?
19. Given $x(t) = sinc(t\ /\ /pi)$, at what times does this function cross zeros?
20. The first zero-crossing of a sinc function occurs at time = B s; 0.5 s; 2 s. what is the bandwidth of each of these three cases?
21. What is the width of the main lobe of the CTFT of a square pulse of widths: T_s, $T_s/2$, $\pi/2$, and 3 s.
22. If the main lobe width of a sinc function (one sided) is equal to $\pi/2$, then how wide is the square pulse in time?
23. What is the CTFT of an impulse train with period equal to 0.5 secs.
24. Convolution in time domain of two sequences represents what in frequency domain?
25. What is the Fourier transform of an impulse of amplitude 2 v in time domain?
26. If a signal is shifted by 2 seconds, what happens to its CTFT?
27. the CTFT of a periodic signal is continuous white the CTFSC is discrete, true or false?
28. If the CTFSC of a signal at a particular harmonic is equal to 1/2, then what is the value obtained via CTFT at the same harmonic?