Discrete Time Signals and Fourier series

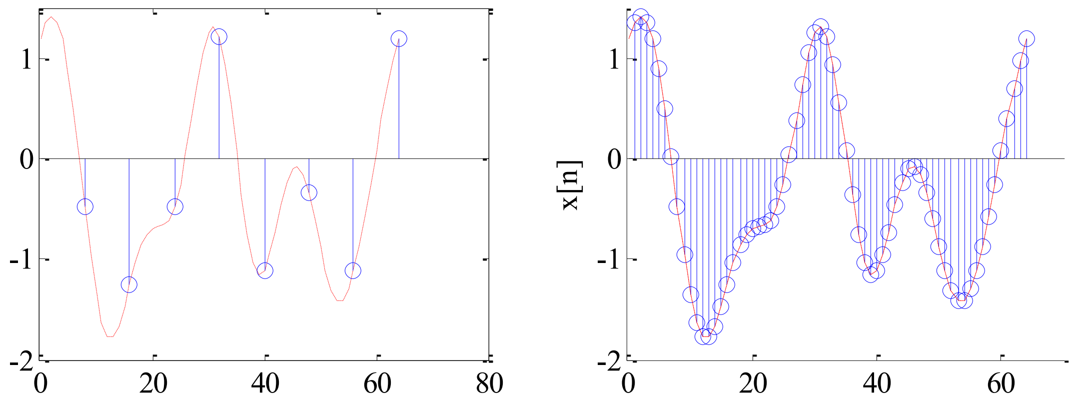
In previous two chapters we discussed the Fourier series for continuous-time signals. We showed that the series is in fact an alternate representation of the signal. This representation can be done in a trigonometric form with sine and cosine functions or with complex exponentials. Both forms are equivalents. Fourier analysis allows us to represent the signal as a weighted sum of harmonic signals. The weights of harmonic can be thought of as the spectrum of the signal. In previous two chapters, our discussion was limited to continuous time signal. In this section we will discuss Fourier series for discrete signals.

Properties of discrete signals

Many of the signals we deal with are sampled analog signals, such as voice, music, and medical/biological signals. This is done by instantaneous sampling of the underlying signal and recording the measured data. A key question facing the engineer is how fast to sample?

Sampling of signals

Suppose we have an analog signal and we wish to create a discrete version of it by sampling it. In Fig. 1, we show an analog signal sampled at two different rates. It is obvious just by looking that the sampling rate chosen in Fig. 1(a), that the rate is not quick enough to capture all the ups and downs of the signal. Some high and low points have been missed. But the rate in Fig.1(b) looks like it might be too fast as it is capturing far more samples than we probably need. So clearly there is an optimum sampling rate which captures enough information without overdoing it such that the underlying analog signal can be described correctly.

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(See Matlab Program 1)
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Figure 1 – Continuous and a discrete signal

This is where we invoke the famous **Sampling Theorem** by Shannon. The theorem says:

> *For any analog signal containing among its frequency contents a maximum frequency of $f_{\max}$, the underlying signal can be represented faithfully by N equally spaced samples, provided the sampling rate is at least two times $f_{\max}$ samples per second.*

So for any signal, a maximum **Sampling Period** that will still allow the signal to be reconstructed from its samples is specified as:

$$T_s = \frac{1}{2f_{\max}} \quad \text{seconds} \tag{1.1}$$

Sampling frequency is specified by the inverse of the sampling period.

$$F_s = \frac{1}{T_s} \quad \text{samples/second} \tag{1.2}$$

The maximum frequency of an analog signal that can be represented unambiguously by a discrete signal with a sampling period of T_s seconds is given by:

$$f_{\max} = \frac{1}{2T_s} Hz$$

$$or \qquad (1.3)$$

$$\omega_{\max} = 2\pi f_{\max} = \frac{\pi}{T_s} radians/\sec$$

Now we will show a property of discrete signals that is most perplexing and causes a great deal of confusion. In Figure 2(a), we show a continuous signal, x(t). We sample this signal at 8 samples per second in Fig. 2(b) and then with 12 samples per second in Fig. 2(c) for a total of 48 samples.

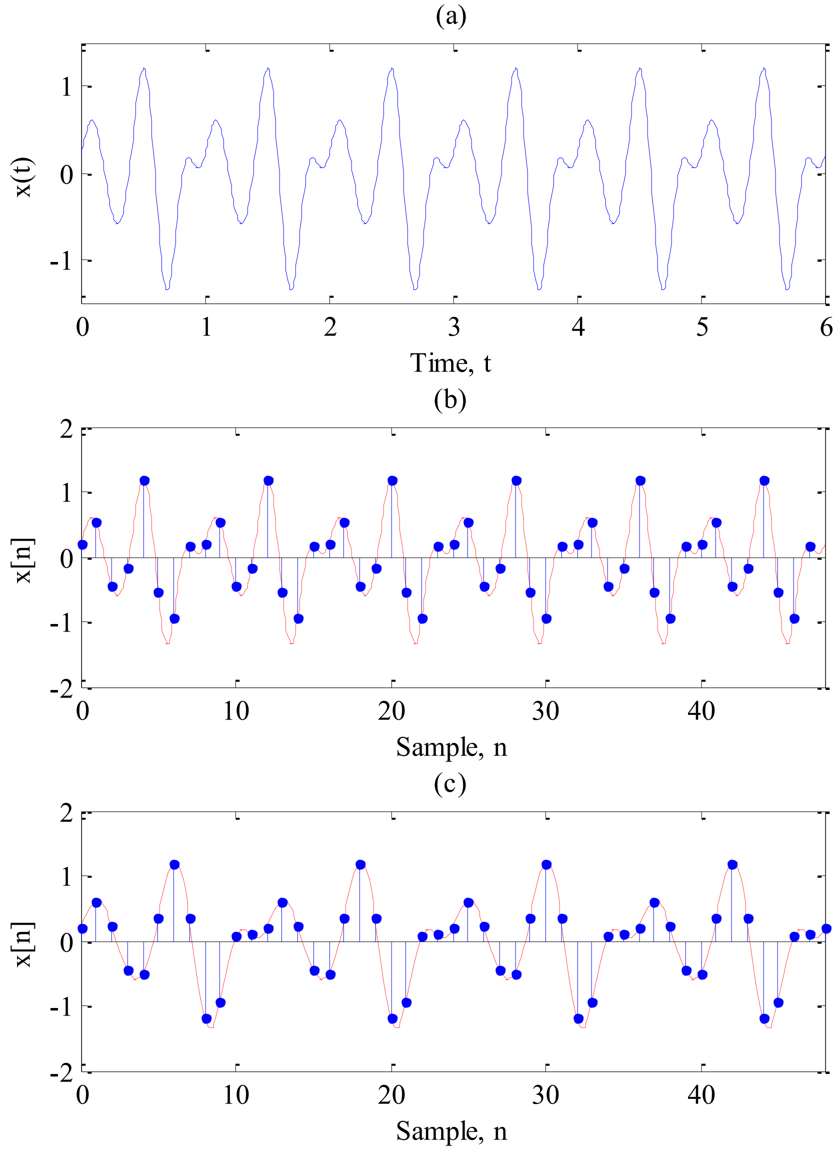

Figure 2 – (a) A continuous signal, (b) sampled at 8 samples per second and (c) sampled at 12 samples per second.

`(See Matlab Program 2)`

The signal is given by the following expression. It contains frequencies 1, 2, 3, and 4 Hz and no others.

$$x(t) = .25\sin(2\pi t) + .7\cos(4\pi t) - .5\cos(6\pi t) + .15\sin(8\pi t)$$

The highest frequency in this signal is 4 Hz. Fig. 3(a) shows the Fourier series coefficients of this continuous signal. The coefficients span from -4 to +4 Hz and are symmetrical about the zero frequency. We can see that the coefficients are 0.125, 0.35, 0.25 , 0.075, for frequencies ± 1, ± 2, ± 3 and ± 4 Hz respectively. As we know, Fourier series coefficients measure the "content" of each frequency and hence the computed coefficients occur only at frequencies actually present in the signal, which are 1, 2, 3, and 4 Hz. This signal contains no other frequencies so all the other coefficients are zero. The spectrum is shown in Fig. 3(a). No confusion here.

Now convert this signal to a discrete signal by sampling it at 12 samples per second. This rate is slightly higher than the minimum sampling rate of 2 times $f_{\max}$ or 8 Hz. The discrete version is given by the expression:

$$x(t) = .25\sin(2\pi k/12) + .7\cos(4\pi k/12) - .5\cos(6\pi k/12) + .15\sin(8\pi k/12)$$

Here k is the sample index, or a point at which the signal is being "discretized". Now without actually going over the process of how this was done, we show the spectrum of this discrete-time signal. In Fig. 3(a), we see the spectrum of the continuous signal. It is limited to ± 4 Hz as would be expected. But we get an odd thing when we compute the Fourier series coefficients of the sampled discrete signals. As shown in Fig. 3(b), instead of finding zero's outside of the actual bandwidth, we get ghosts-like copies of the same spectrum centered at integer multiple of the sampling frequency. Content appears at frequencies that are not present in the signal.

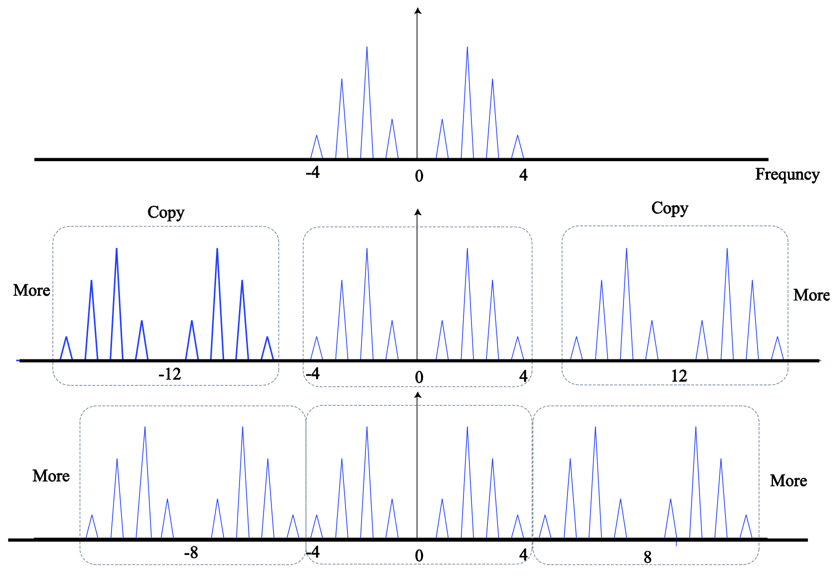

Figure 3 – Replicating spectrums as a consequence of sampling

The spectrum around zero frequency (the center part in Fig. 3(a)) repeats at the sampling frequency of the sampled signal. Every 12 Hz, there is a copy of the spectrum. These replications occur every $\pm 12, \pm 24, \cdots$ endlessly. In Fig. 3(c) we show the same signal sampled at 8 Hz, and now the spectrum repeats at $\pm 8, \pm 16, \cdots$. If we sampled the continuous signal at a rate less than 8 Hz, which is the minimum required by the sampling theorem, the spectrum would begin to overlap and that is a problem which we will discuss in Chapter 6. This overlapping is called aliasing, resulting from a sampling rate that is less than twice the maximum frequency in the signal or also called the **Nyquist rate**.

This replicating of the spectrum is a consequence of discrete sampling. We certainly do not see it in the spectrum of a continuous-time signal. In this section, we are going to discuss the Fourier series representation of discrete signals, calculation of the series coefficients and we are going to talk about why this spectrum replication happens.

Specifying a discrete signal

If a continuous signal is referred to as x(t) then a discrete sampled signal is written as

$$x(kT_s), \quad k = 0, \pm 1, \pm 2, \cdots$$

Where T_s is the sampling period, or the time between any two successive samples. The index k is called the sample number. The quantity kT_s is a measure of time.

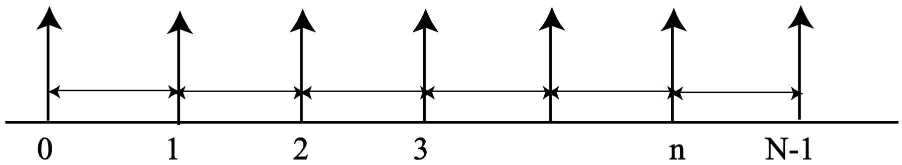

Figure 4 – Discrete signal samples

To create a discrete signal from a continuous signal, we take two steps, first the continuous signal is multiplied by an impulse train of the sampling period. But since mathematically this is still a continuous signal, we then multiply the sampled signal once again by an impulse train, point by point. The sum of all of those points is the discrete signal.

$$\begin{aligned} x(kT_s) &= x(t) \times \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \\ x[k] &= \sum_{k=-\infty}^{\infty} x(kT_s)\delta(t - kT_s) \end{aligned} \tag{1.4}$$

The term $x(kT_s)$ is considered continuous while the term x[k] is its discrete equivalent. It's clear what is happening here, the delta function is "combing" the signal to create a discrete signal. (We use the square brackets [ ] to denote a discrete sequence and use regular brackets ( ) for a continuous signal.)

Let's take a sine wave and plot its continuous and discrete versions.

$$x(t) = \sin(2\pi f_0 t), \quad f_0 = 1.$$

We replace continuous time t with kT_s and compute a few values of the discrete signal as follows. These are plotted in Fig. 5b.

$$\begin{aligned}
&x[-10] = \sin[2\pi(-10 \div 5)] = 0 \\
&x[-9] = \sin[2\pi(-9 \div 5)] = \sin[-3.6\pi] = .951 \\
&x[-8] = \sin[2\pi(-8 \div 5)] = \sin[-3.2\pi] = .588 \\
&x[-7] = \sin[2\pi(-6 \div 5)] = \sin[-2.4\pi] = -.951 \\
&x[-6] = \sin[2\pi(-5 \div 5)] = \sin[-2\pi] = 0 \\
&x[-5] = \sin[2\pi(-4 \div 5)] = \sin[-1.6\pi] = .951
\end{aligned}$$

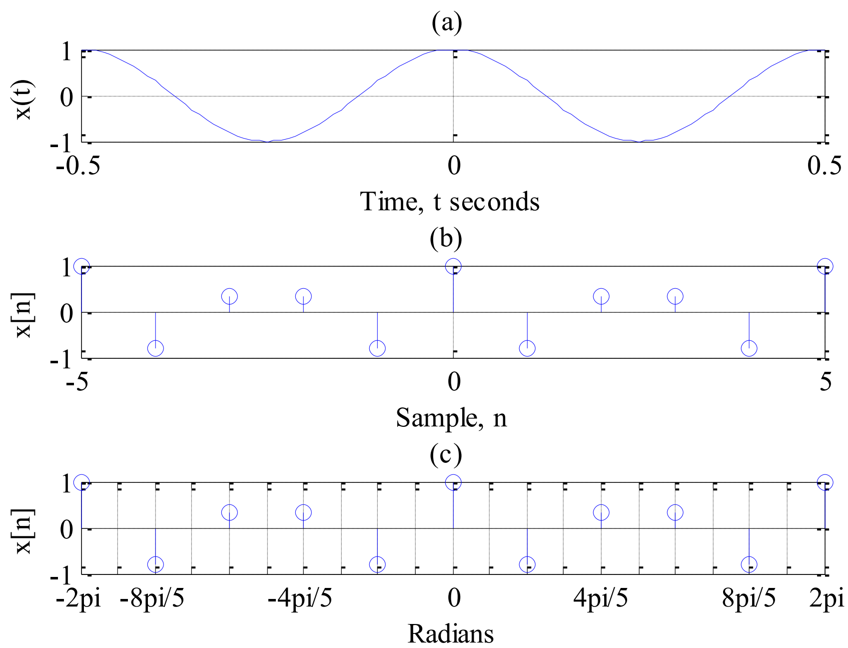

Figure 5 – Sampling of a continuous signal

(Matlab Program 3)

Discrete signal representation

There are two ways to specify a sampled signal. One is by sample numbers. In Figure 5, we show two periods of the signal. The signal covers two cycles in 2 seconds. Each cycle is sampled with five samples, so we have a total of ten samples in Fig. 5(b). This is the discrete representation of signal x[k] in terms of samples. The rate of sampling is 5 samples per second or 5 samples per cycle. This is a common way of showing a discrete signal particularly if the signal is not periodic.

As we will see, there are advantages in specifying the signal by its phase. In polar form, a periodic signal is said to cover 2π radians in one cycle. We can replace the sample with its phase value for an alternate way of describing a discrete signal. There are five samples over each 2π or equivalently a discrete angular frequency of $2\pi/5$ radians. Each sample moves the signal further in phase by $2\pi/5$ radians from the previous sample, with two cycles or 10 samples covering 4π radians as in Fig. 5(c).

Parameters of a discrete signal

To create a discrete signal we sample this signal with sample time of T_s. The sample time as we said should be small enough to capture all important information. The

Sampling theorem tells us that it should be no larger than $1/2f_{\max}$. The sampling period T_s is independent of the fundamental period T_0 of the continuous signal and can be any number smaller than the Nyquist threshold.

T_0 – the fundamental period of the continuous signal
T_s – the sampling period of the discrete signal

If we know the signal frequency f_0 and the sampling frequency, we can also write the signal this way replacing T_s with $1/F_s$.

$$x[k] = \sin\left[\left(\frac{2\pi f_0}{F_s}\right)k\right]$$

Let's give the term inside the parenthesis a special name, calling it **Digital frequency**.

$$\Omega_0 = \frac{2\pi f_0}{F_s} \quad \frac{radians \times cycles/\sec ond}{sample/\sec ond} \rightarrow \frac{radians}{sample} \tag{1.5}$$

The units of this "frequency" are given as radians per sample and not as radians per second. So it is not really a frequency, but we call it that for lack of a better name. You can also think of it as "phase advance".

If we have a signal of frequency 10 Hz and we sample it with a sampling frequency F_s = 30 Hz, then its digital frequency is equal to $(2/3)\pi$. What does this number mean? It means that each sample moves the signal by this many radians. If a cycle contains 2π radians, and each sample covers $(2/3)\pi$ radians, then it will take 3 samples to complete a cycle. The digital frequency in this case is:

$$\Omega_0 = \frac{2\pi \times 10}{30} = 2\pi/3$$

K_0, the period can be seen as a ratio of the sampling frequency to the fundamental frequency or the maximum frequency:

$$K_0 = \frac{F_s}{f_0} = \frac{30}{10} = 3$$

We want this ratio to be bigger than 2 and usually much bigger than that. The fundamental period K_0 is an integer and represents number of samples after which the signal starts repeating again. If there are K_0 samples in one period, then K_0 times the digital frequency Ω must equal 2π. (Note the units of this frequency are radians per sample.) We define the period of the signal in samples as:

$$K_0\Omega_0 = 2\pi \;\; or \;\; K_0 = \frac{2\pi}{\Omega_0} \tag{1.6}$$

The fundamental frequency is given by:

$$\Omega_0 = \frac{2\pi}{K_0} \tag{1.7}$$

The period of the digital frequency is always 2π because that condition is part of its definition. The smaller the digital frequency, more samples are needed to complete one cycle. Having K_0 in the denominator says exactly the same thing.

If the underlying signal is periodic, is the sampled discrete signal also periodic? Not necessarily. It will be if and only if N is an integer.

$$N = mK_0 = m\frac{T_0}{T_s} \tag{1.8}$$

The fundamental period of the continuous signal is an integer multiple of the ratio between the fundamental time and the sampling time. This is also same as saying

$$N = mK_0 = m\frac{F_s}{f_0} \tag{1.9}$$

For example a signal with fundamental frequency of 5 which is sampled at a rate of 20, will have a fundamental period of 4 and as such this sampling rate would result in a discrete signal that is periodic.

Discrete signal periodicity

The definition of a periodic signal for a discrete signal is the same as for the continuous case, which is

$$x[k] = x[k + K_0] \tag{1.10}$$

The discrete values of the signal repeat every N samples. It may seem like a trivial exercise, but we are now going to look at the periodicity of a discrete sinusoid $\sin(\Omega n)$.

The digital frequency Ω here is a general frequency term. We use the subscript Ω_0 whenever we are talking about the fundamental frequency and just Ω when we are talking about any other discrete harmonic frequency.

How can we show that a discrete signal $\sin(\Omega n)$ is periodic? Using the above expression in Eq. (1.8) to write:

$$\begin{aligned}\underline{\sin(k\Omega)} &= \sin(\Omega(k+K_0)) \\ &= \underline{\sin(k\Omega + K_0\Omega)}\end{aligned}$$

The equation holds under only one condition. That is if

$$K_0\Omega = 2\pi k \tag{1.11}$$

That's because at even multiples of π, the value of sine is zero. All integer values of k satisfy the condition.

With discrete signals, there is natural ambiguity about what frequency a set of discrete samples represent. In Figure 6 we see two different continuous signals, one a cosine of frequency 2 Hz and the other of frequency 10 Hz, both of very different frequencies yet they map to the same discrete samples. In fact the same samples would fit many other harmonics! This creates an ambiguity in discrete signal processing that we don't have in continuous signals. When we look at a discrete signal, we truly don't know what frequency it is supposed to represents. An infinity of signals fit the same points. So intuitively, replication of the spectrum is saying exactly that. Since the algorithm (DFT, FFT etc.) does not know which frequency is the actual one in the signal, it just repeats the same spectrum at all possible harmonics. Basically saying, I don't know which one is correct, you pick!

We do of course have some idea of about the target signal frequencies. The sampling frequency is carefully selected to capture the underlying signal. We are usually not interested in frequencies outside of a certain range. Since each cycle contains exactly the same information, we can just limit the analysis to one cycle, ignoring all the others replications as truly imaginary! For this reason, the Fourier analysis to determine the coefficients of a discrete signal can be limited to just one cycle. This is why we like to represent the signal by its phase where a 2π range makes more sense than dealing with numbered samples. Any one period contains all available information about the periodic signal.

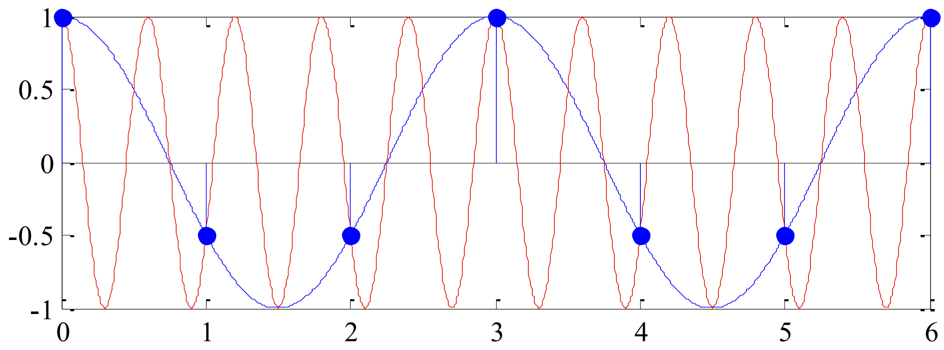

Figure 6 – A discrete signal can represent any number of continuous signals.

(Matlab Program 4)

Why are we bothering with the periodicity of a discrete signal? Aren't all sinusoids periodic by definition? Yes, that is true for the continuous case but not always true for the discrete sinusoids.

Example 3-1

What is the digital frequency of this signal? What is its period?

$$x[k] = \cos\left[\frac{2\pi}{3}k + \frac{\pi}{3}\right]$$

The digital frequency Ω_0 of this signal is $\frac{2\pi}{3}$. Its period K_0 is equal to 3. The values of x[k] repeat after every three samples as can be seen in Fig 7.

$$K_0 = \frac{2\pi}{2\pi/3} = 3$$

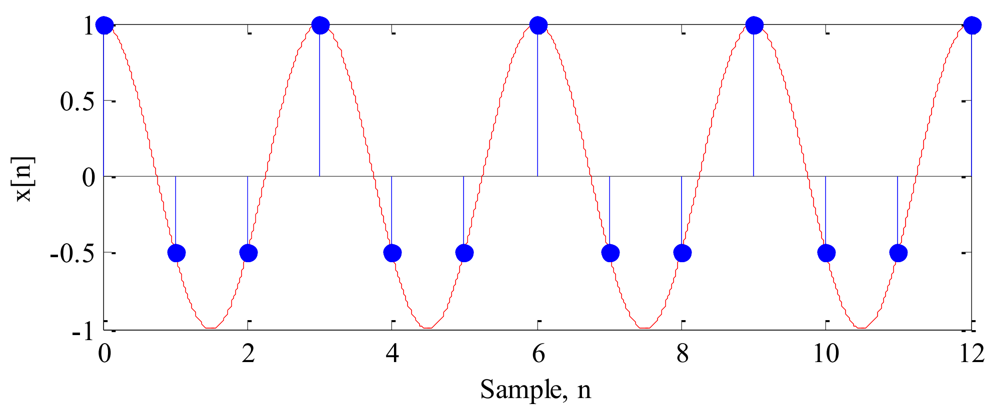

Figure 7 – Discrete signal of example 3-1

(Matlab Program 5)

Example 3-2

What is the period of this discrete signal? Is it periodic?

$$f[k] = \sin\left(\frac{3\pi k}{4} + \frac{\pi}{4}\right)$$

The digital frequency of this signal is equal to $\Omega_0 = 3\pi/4$. The period of the signal is:

$$3\pi m = m4$$
$$K_0 = 8, \; m = 2$$

The fundamental period of the signal equals 8, because that is the minimum number of samples needed to achieve an integer multiple of 2π. So the signals repeats after every 6π radians.

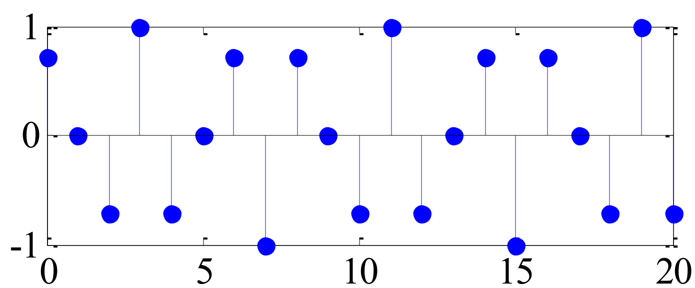

Figure 8 – Discrete signal of Example 3-2

Example 3-3

Is this discrete signal periodic?

$$f[n] = \sin\left[.5n + \pi\right]$$

The fundamental frequency of this signal is 0.5.

$$K_0 = \frac{2\pi}{\Omega_0} k = 12\pi k$$

Can we call this the period of the signal? Yes, but it is not a rational number (a ratio of integers), hence it can never result in a repeating signal. The continuous signal as shown here is of course periodic but we don't see any periodicity in the discrete samples.

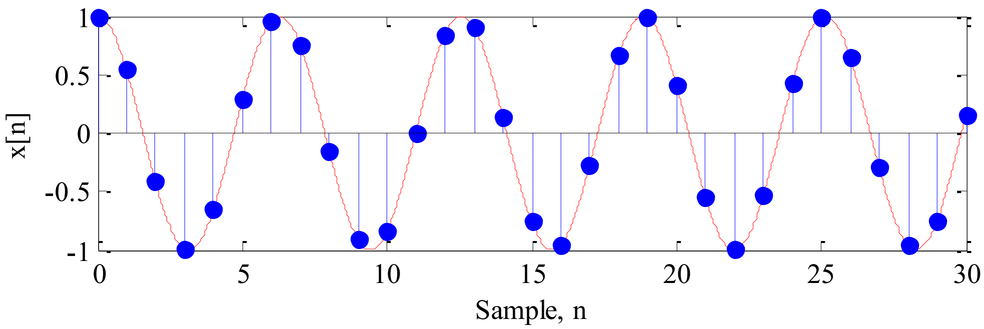

Figure 9 – Non-periodic discrete signal of example 3-3.
`(Matlab Program 6)`

Basis functions for Discrete-time Fourier series

The continuous-time Fourier series (CTFS) is written in terms of complex exponentials. Because they are harmonic and hence orthogonal to each other, these complex exponentials form a basis set. The series coefficients can be seen as the projection of the signal on to these basis functions. We are now going to develop a Fourier series representation for discrete time signals using as basis functions the discrete time complex exponential. Let's examine the discrete-time exponential and see how its periodicity is affected by taking it into the discrete realm.

The <u>discrete</u> complex exponential is written by replacing t with k. We can write this in terms of the digital frequency as:

$$\begin{array}{ll} e^{j\omega t} & \text{continuous signal} \\ e^{j\Omega k} & \text{discrete signal} \end{array} \tag{1.12}$$

Note that units of both ωt and Ωk are radians. In continuous case, a harmonic is an integer multiple of the frequency. In the discrete case, the harmonic relationship is based on phase:

$$\Omega + 2\pi k \rightarrow \Omega \tag{1.13}$$

The digital frequency Ω is in radians per sample, so clearly the next harmonic frequency is obtained by incrementing Ω by 2π, so that Ω and $(\Omega + 2\pi k)$ are harmonics for all integer k. This is same as saying that ω_0 and $k\omega_0$ are harmonically related for all k.

Every time Ω increases by 2π, we get a new complex exponential given by:

$$e^{j(\Omega+2\pi)k} = e^{j\Omega n}\underline{e^{j2\pi k}} = e^{j\Omega k} \tag{1.14}$$

The second term, $e^{j2\pi k}$ is equal to 1 because:

$$\begin{aligned} e^{j2\pi k} &= \cos(2\pi k) - \underbrace{j\sin(2\pi k)}_{=0} \\ &= \cos(2\pi k) \\ &= 1 \end{aligned}$$

This is quite an interesting result. The exponential $e^{j(\Omega+2\pi)k}$ is exactly the same as the exponential $e^{j\Omega k}$. Although in the continuous case, each and every harmonic different, all harmonics of a discrete signal are exactly the same. To some extent this takes the thunder out of doing Fourier series analysis on a discrete signal. Harmonics do not seem to form a useful basis set.

Example 3-4

Show harmonics of the exponential $e^{j\frac{2\pi}{3}t}$ if it is being sampled with sampling period of 0.25 seconds.

We can write the exponential in discrete form by replacing t with $kT_s = k/4$.

$$y[k] = e^{j\frac{2\pi}{12}k}$$

Let’s plot this signals along with its next two harmonics, which are

$$e^{j\left(\frac{2\pi}{12}+2\pi\right)k} \text{ and } e^{j\left(\frac{2\pi}{12}+4\pi\right)k}$$

We plot only the real part in Fig 10. Why is there only one plot? Simply because the three signals are identical.

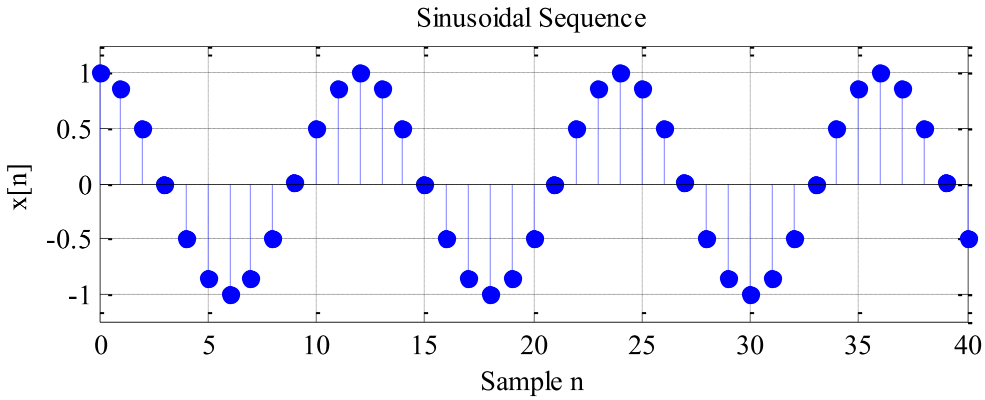

Figure 10 – Three discrete harmonic signals

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(Matlab Program 7)
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What this tells us is that for discrete signal the traditional concept of harmonic frequencies does not lead to anything meaningful. All harmonics are the same. But then how can we do Fourier series analysis on a discrete signal if there are no orthogonal basis signals to represent the analysis signal?

There is, as it turns out, available a set of orthogonal basis set that we can use. So far we only looked at harmonics that differ by phase of 2π. Although they are harmonic in a mathematical sense, these are pretty much useless in a practical sense. Instead of looking very 2π for a harmonic, we need to look elsewhere. We will now reveal a secret: we find hidden inside the 0 to 2π range, there exists an another orthogonal basis set.

Let's see what happens as digital frequency Ω is varied just within the 0 to 2π range. Take the signal $x[k]=e^{j\frac{2\pi}{6}k}$. Its digital frequency is equal $\dfrac{2\pi}{6}$ and its period K_0 is equal to 6. We now know that the signals of digital frequencies $2\pi/6$ and $4\pi/6$ are exactly the same, but what about in between?

We will increase the frequency of this signal in 6 steps, each time increasing it by $2\pi/6$ so that after 6 steps, the total increase will be 2π. We can start with zero frequency, as it makes no difference where you start.

$$
\begin{aligned}
\phi_0 &= 2\pi(n=0)/6 = 0 \\
\phi_1 &= 2\pi(n=1)/6 = 2\pi/6 \\
\phi_2 &= 2\pi(n=2)/6 = 4\pi/6 \\
&\vdots \\
\phi_5 &= 2\pi(n=5)/6 = 10\pi/6
\end{aligned}
$$

The variable n steps from 0 to $K_0 - 1$. There are N harmonics, and we index them with letter n. Index k remains the index of the sample.

In Figure 11, we have plotted the analog signal and the discrete version of the same signal. The discrete frequency appears to increase (more oscillations) at first but then after 3 steps (half of the period, N) it starts to back down again. The discrete signals for frequencies $2\pi/3$ and the $4\pi/3$ appear identical. Reaching the next harmonic at 2π, the discrete signal is back to where it started. Further increases will repeat the same cycle.

So here we have a significant difference in how discrete and continuous signals of same frequencies behave. The analog signals are harmonic along the frequency axis whenever $\omega_k = k\omega_0$. Discrete signals are harmonic in between these values when specified in terms of phase. How do we know these signals are harmonic? The plot of the analog signal at these samples in Fig. 11 tells us that they are harmonic. Of course, if we can do the orthogonality test, and we find that they are indeed harmonic to each other.

$$\sum_{n=0}^{K_0-1} \phi_1 \phi_2^* = 0$$

These 6 signals, which we refer to by K_0, are an orthogonal basis set that can be used to represent a discrete signals. The weighted sum of these K_0 special signals is the discrete Fourier series representation of the signal. Unlike the continuous signal, here the meaningful range is limited to a finite number of harmonics only.

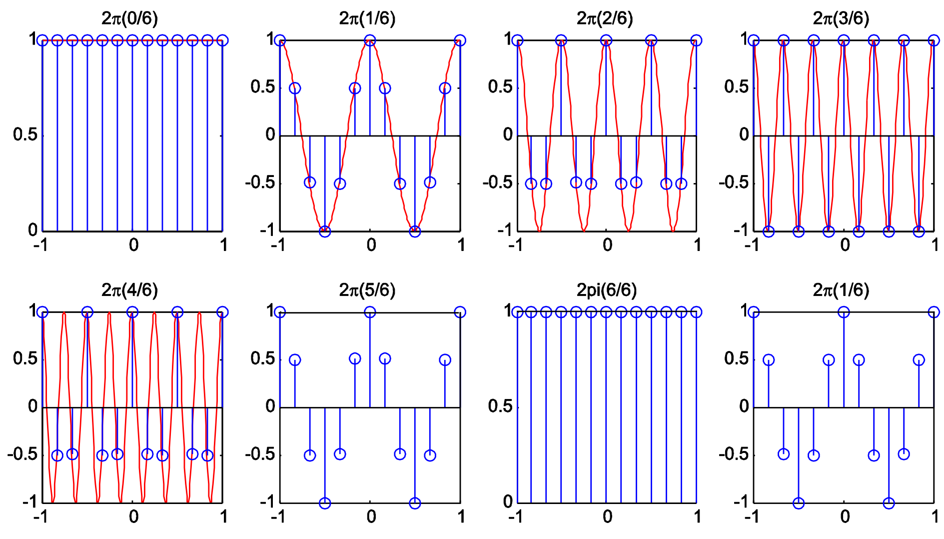

Figure 11 – Discrete signals in between harmonic frequencies

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Discrete Time Fourier Series - DTFS

We note key ideas about discrete signals.

1. We do not know the underlying analog signal nor its frequency. All we know is that if the sampling frequency is Fs, then we can from a discrete signal unambiguously extract signals of only of frequency half as much.
2. A discrete signal is defined by its digital frequency. The units of digital frequency are in radians per sample. We can think of it as being defined over a circle from 0 to 2π (or $-\pi$ to $+\pi$).
3. A discrete signal of frequency Ω_0 is exactly the same as all its harmonics when $\Omega_0 \pm 2\pi k$ for all k.
4. There are only N distinct discrete–time complex exponential signals that are harmonically related for any given period N. K_0 is the smallest such number and called the fundamental period.

The Discrete Time Fourier series (DTFS) is the discrete representation of a periodic signal by a linear weighted combination of N complex exponentials. These orthogonal exponentials exist within just one cycle. Note there are only N of these, not an infinite number as in continuous time Fourier series representation. The frequency decomposition here is discrete just as it is for the Continuous Time Fourier series (CTFS). The coefficients describe the content of each basis function in the signal. One can also think of the DTFS coefficients (DTFSC) as a correlation of the complex exponential with the target signal. These coefficients form a discrete signal hence the spectrum of a discrete-time periodic signal is also discrete.

If $\Omega_0 = \frac{2\pi}{5}$, we can write the discrete harmonic complex exponentials as:

$$e^{-j\ \Omega_0 n\ k},$$
$$e^{-j\left(\frac{2\pi}{n}\times 0\right)k}, e^{-j\left(\frac{2\pi}{n}\times 1\right)k}, e^{-j\left(\frac{2\pi}{n}\times 2\right)k}, e^{-j\left(\frac{2\pi}{n}\times 3\right)k}, \cdots, e^{-j\left(\frac{2\pi}{n}\times (K_0-1)\right)k}$$

The index n is used to indicate the harmonics. The index k is the time sample. There seem to be many different ways people write the exponent of the complex exponential. I like to keep the fundamental frequency and its index together, and k the time index at the end. Many sources also use n and k in opposite sense from the way I have used them. Some use n for time index and k for harmonic index. The indexes can be confusing because both the number of harmonics (referred to by index n) and the numbers of samples (referred to by k) are same and equal to the number K_0. The discrete-time representation of the signal is written as the weighted sum of these.

$$x[k] = \sum_{k=0}^{K_0-1} C_n \, e^{j\,(n\,\Omega_0)k} \tag{1.15}$$

The complex coefficients, $C_0, C_1, \cdots, C_{(K_0-1)}$ are given as

$$C_n = \frac{1}{K_0} \sum_{n=0}^{K_0-1} x[k] \; e^{-j\,(n\,\Omega_0)k} \tag{1.16}$$

Note that we are summing over just one period. So (1.18) says that a discrete Fourier series is a decomposition of a single period of the signal into a fundamental digital frequency, and K_0-1 harmonics of that frequency.

We said that all discrete harmonics are the same, and now we show that the DTFS coefficients of the nth harmonic are exactly the same as the coefficient for a harmonic that is an integer multiple of mK_0 samples away so that:

$$C_n = C_{(n+mK_0)} \tag{1.17}$$

Here m is an integer. The nth coefficient is equal to

$$C_n = \sum_{k=0}^{N-1} x[n] e^{-j\;n\Omega_0\;k}$$

The $(n + mK_0)$ coefficient is given by

$$C_{(n+mK_0)} = \sum_{k=0}^{N-1} x[k] e^{-j(n+mK_0)\Omega_0 k}$$

$$= \sum_{k=0}^{N-1} x[k] e^{-jn\Omega_0 k} e^{-jmK_0\Omega_0 k}$$

The second part $e^{-jmK_0\Omega_0 k} = e^{-jm2\pi k}$ is equal to 1. (Because the value of the complex exponential at integer multiples of 2π is 1.0) So we have:

$$C_{(n+mK_0)} = \sum_{k=0}^{N-1} x[k] e^{-jn\Omega_0 k}$$

$$= C_n$$

So indeed the coefficients repeat for a discrete-time periodic signal. In practical sense, this means we can limit the computation to just K_0 harmonics.

In earlier section, we stated that the discrete Fourier analysis results in replicating spectrums. This is a very different situation from the case of continuous signals, which do not have such behavior. Discrete signals do this because as we allow n to vary over all values of digital frequency, which are repeating every K_0 samples, we can no longer tell the harmonics apart so we are computing the same K_0 numbers over and over again.

Example 3-5

Find the discrete time Fourier series coefficients of this signal.

$$x[k] = 1 + \sin\left(\frac{2\pi}{10}k\right)$$

The fundamental period of this signal is, $K_0 = 10$ as we can see.

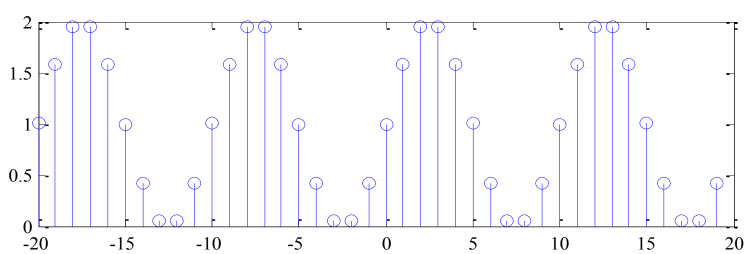

Figure 12 – Signal of example 3-5

Now write the Euler equivalent expression for this signal, we get

$$x[k] = 1 + \frac{1}{2j} e^{j\left(\frac{2\pi}{10}\right)k} - \frac{1}{2j} e^{-j\left(\frac{2\pi}{10}\right)k}$$

From this expression, we can compute the DTFSC from C_0 to C_9 by setting n, from 0 to 9. We get the following result. Note that because the analysis signal has only two frequencies, corresponding to index n = 0, which is the zero frequency and n = 1, which corresponds to the fundamental frequency the coefficients for remaining harmonics are zero. We can write the coefficients as

$$C_n = \frac{1}{K_0}\sum_{n=0}^{K_0-1} x[k]\ e^{-j(n\Omega_0)k}$$
$$= \frac{1}{10}\sum_{n=0}^{9} x[k]\ e^{-j\left(\frac{2\pi}{10}n\right)k}$$

x[k]= $[1.0000, 1.5878, 1.9511, 1.9511, 1.5878, 1.0000, 0.4122, 0.0489, 0.0489, 0.4122]$

$$C_n = C_0 = \frac{1}{10}\sum_{k=0}^{9} x[k]\underbrace{e^{-j\ o\Omega_0\ k}}_{1} = \frac{1}{10}\sum_{k=0}^{9} x[k]\ = 1$$

In computing the next coefficient, we compute the value of the complex exponential for n = 1 and then for each value of k, we use the corresponding x[k] and the value of complex exponential. The summation will give us these values.

$$C_1 = \frac{1}{10}\sum_{k=0}^{9} x[k]e^{-j\ 1\cdot\Omega_0\ k} = \frac{1}{2j}$$
$$C_{-1} = \frac{1}{10}\sum_{k=0}^{9} x[k]e^{j\ 1\cdot\Omega_0\ k} = -\frac{1}{2j}$$

Of course, we can see the coefficients directly in the complex exponential form of the signal. The rest of the coefficients from C_2 to C_9 are zero. However, the coefficients repeat after C_9 so that $C_{(1+9k)} = C_1$.

Example 3-6

Compute the DTFSC of this discrete signal.

$$x[k] = 0.5 + 0.25\cos\left(\frac{2\pi}{5}k\right) - 0.6\sin\left(\frac{2\pi}{4}k\right)$$

The period, K_0 of the cosine is 5 and the period, K_0 of sine is 4. Period of the whole signal is 20 because it is the least common multiple of 4 and 5. This signal repeats after every 20 samples. In time domain, we have highlighted the periodic section of the 20 samples.

The fundamental frequency of this signal is

$$\Omega_0 = \frac{2\pi}{20} = \frac{\pi}{10}$$

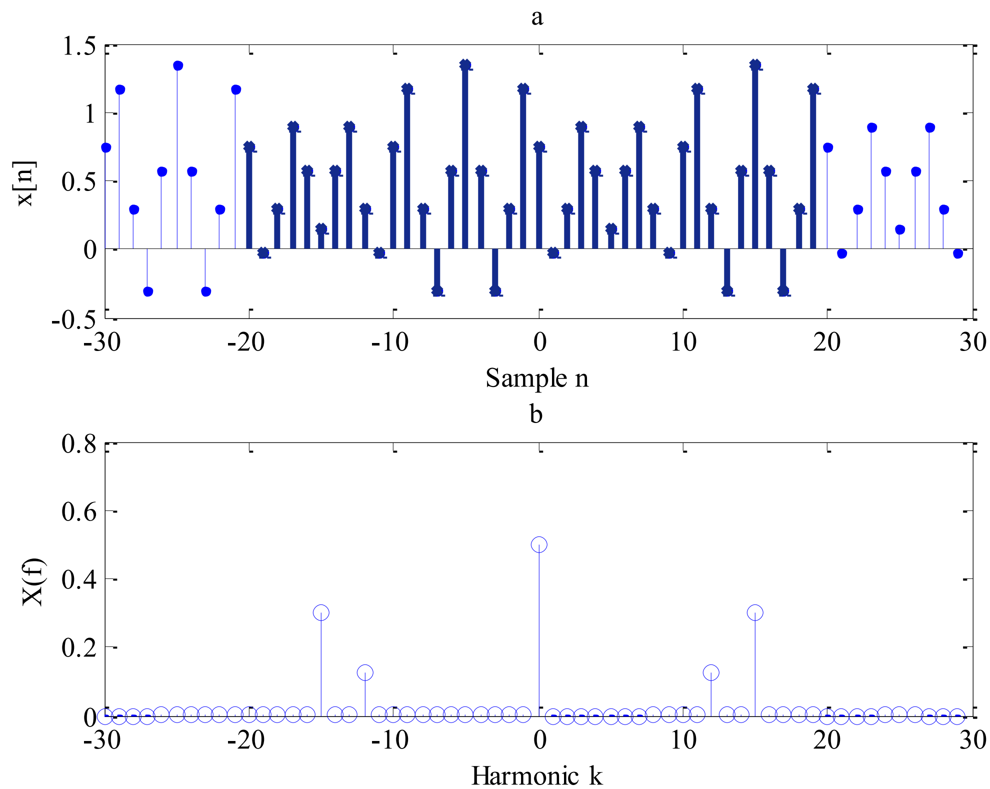

Figure 13- Signal of example 3-6

$$C_n = \frac{1}{K_0} \sum_{n=0}^{K_0-1} x[k]\ e^{-j(n\Omega_0)k}$$

$$= \frac{1}{20} \sum_{n=0}^{19} x[k]\ e^{-j\left(\frac{\pi}{10}n\right)k}$$

$$x[k] = \frac{1}{20}\left[x[0]e^{-j(2\pi/5)}\right]$$

The Fourier coefficients repeat with a period of 20. Each complex exponential will vary in digital frequency by $\pi/10$. Based on this knowledge, when we look at the above expansion, we can see that $(2\pi/4)$ exponential falls at n = 5. $(5\times(\pi/10)=2\pi/4)$ and exponential $(2\pi/5)$ falls at n = 4 $(4\times(\pi/10)=2\pi/5)$. Also note that the Fourier series is a breakdown of the signal in sinusoids. But here our target signal is conveniently already in sinusoids. So all we have to do to find the coefficients it to just write it out in the Euler formulation and then pick out the coefficients by inspection. We did several examples of this process in Chapter 2.

We can write this signal as

$$x[k]=0.5+0.125\left(e^{\left(\frac{2\pi}{5}k\right)}+e^{-\left(\frac{2\pi}{5}k\right)}\right)+0.3j\left(e^{\left(\frac{2\pi}{4}k\right)}-e^{-\left(\frac{2\pi}{4}k\right)}\right)$$

From here, we see that the zero-frequency harmonic has a coefficient of 0.5. The frequency $(\pm 2\pi/5)$ has coefficients of .125 and so forth.

$$\begin{aligned}&C_0=C_n=0.5\\&C_4=.125 \quad C_{-4}=.125\\&C_5=.3j \quad C_{-5}=-.3j\end{aligned}$$

Working with sines and cosines is almost trivial, because we already know what is "in" the signal by looking at the equation. In example 3-7 we will look at a signal where the coefficient computation using closed form equations is not so simple.

Example 3-7

Compute the DTFS of this periodic discrete signal. The signal repeats with period 4 and has two impulses of amplitude 2 and 1.

The fundamental frequency of this signal is

$$\Omega_0=\frac{2\pi}{4}=\frac{\pi}{2}$$

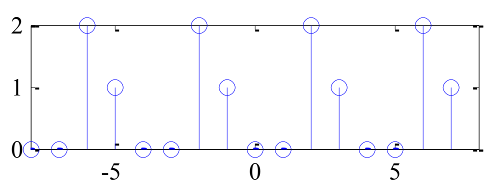

Figure 14 – Signal of example 3-7

We write the expression for the DTFSC from Eq.(1.18)

$$C_n = \sum_{k=0}^{3} x[k] e^{-j\left(\frac{\pi}{2}n\right)k}$$

To solve this summation in closed form is the hard part. In nearly all such problems we need to know series summations or the equation has to be solved numerically. In this case we do know the relationship. We first express the complex exponential in its Euler form. We know values of the complex exponential for argument $\pi/4$ are 0 and 1 respectively for the cosine and sine. We write it in a concise way as:

$$e^{-j\left(\frac{\pi}{2}\right)nk} = \left(\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} - j\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}\right)^{kn} = (-j)^{nk}$$

Now substitute this into the DTFSC equation and calculate the coefficients, knowing there are only n = 4 harmonics in the signal because the number of harmonics are equal to the fundamental period of the signal.

$$C_n = \frac{1}{4}\sum_{k=0}^{3} x[k](-j)^{kn}$$

Since we are interested in the harmonics, start with n = 0, and then multiply each $x[k]$ with $(-j)^{k(n=0)}$ to get the following results. The go to n = 1 and repeat the process.

$$\begin{aligned}
C_0 &= \frac{1}{4}\; 2+1 \;= \frac{3}{4} && \mathit{for}\ n=0,\ k=0,1,2,3 \\
C_1 &= \frac{1}{4}(2-j1) = \frac{1}{2} - \frac{j}{4} && \mathit{for}\ n=1,\ k=0,1,2,3 \\
C_2 &= \frac{1}{4}\; 2-1 \;= \frac{1}{4} && \mathit{for}\ n=2,\ k=0,1,2,3 \\
C_3 &= \frac{1}{4}\; 2+j1 \;= \frac{1}{2} + \frac{j}{4} && \mathit{for}\ n=2,\ k=0,1,2,3
\end{aligned}$$

We can setup the DTFSC equation in matrix form by setting the basic exponential to a constant and then writing it in terms of two variables, the index n and k.

$$\begin{aligned}
e^{j\Omega_0} &= W \\
e^{-jn\Omega_0 k} &= W^{-nk}
\end{aligned}$$

Now we write

$$C_n = \frac{1}{K_0} x[k] \begin{bmatrix} W^{-0\times 0} & W^{-1\times 0} & W^{-2\times 0} & W^{-3\times 0} \\ W^{-0\times 1} & W^{-1\times 1} & W^{-2\times 1} & W^{-3\times 1} \\ W^{-0\times 2} & W^{-1\times 2} & W^{-2\times 2} & W^{-3\times 2} \\ W^{-0\times 3} & W^{-1\times 3} & W^{-2\times 3} & W^{-3\times 3} \end{bmatrix}_{n\times k}$$

Here the each column represents the harmonic index n and each row the time index, k. It takes 16 exponentiations, 16 multiplications and four summations to solve this equation. We will come back to this matrix methodology again when we talk about DFT and FFT in Chapter 5.

We used Matlab to compute the coefficients. Here is what we get. Same as the closed form solution.

$$\begin{bmatrix} 2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} = \begin{bmatrix} .75 & .5-.25j & .25 & .5+.25j \end{bmatrix}$$

Matlab Program 11

Example 3-8

Find the discrete-time Fourier series coefficients of this signal. This signal is part of an important class of signals that are similar to square pulses. They are even harder to solve using closed form solution.

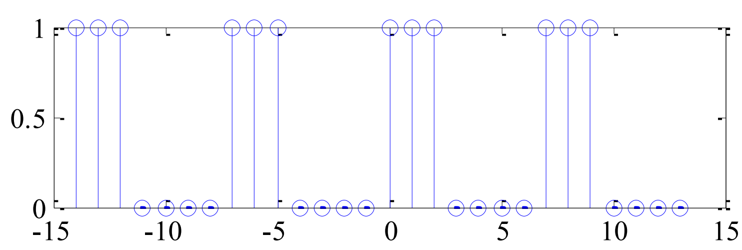

Figure 15 – Signal of example 3-8 with N = 3, K0 = 7

We split the summation in two parts.

$$C_n = \frac{1}{K_0}\sum_{k=-N}^{N} 1\cdot e^{-jn\Omega_0 k} + \frac{1}{K_0}\sum_{k=N+1}^{K_0-N-1} 0\cdot e^{-jn\Omega_0 k}$$

$$= \frac{1}{K_0}\left[e^{-jn\Omega_0 N}\,\frac{1-e^{-jn\Omega_0(2N+1)}}{1-e^{-jn\Omega_0}}\right]$$

This can be simplified (using series summation formulas) to this form.

$$C_n = \frac{1}{K_0}\left[\frac{\sin\ Nn\pi / K_0}{\sin\ n\pi / K_0}\right] \tag{1.18}$$

This function looks lot like a sinc function but actually is a function called Diric. To draw the graph, we assume N = 3 and K_0 = 7 and K_0 =15 and 25. Note as the signal spreads, the components get more numerous. We will come to this property in the next section when we talk about aperiodic signals and Fourier transform.

Matlab program 12

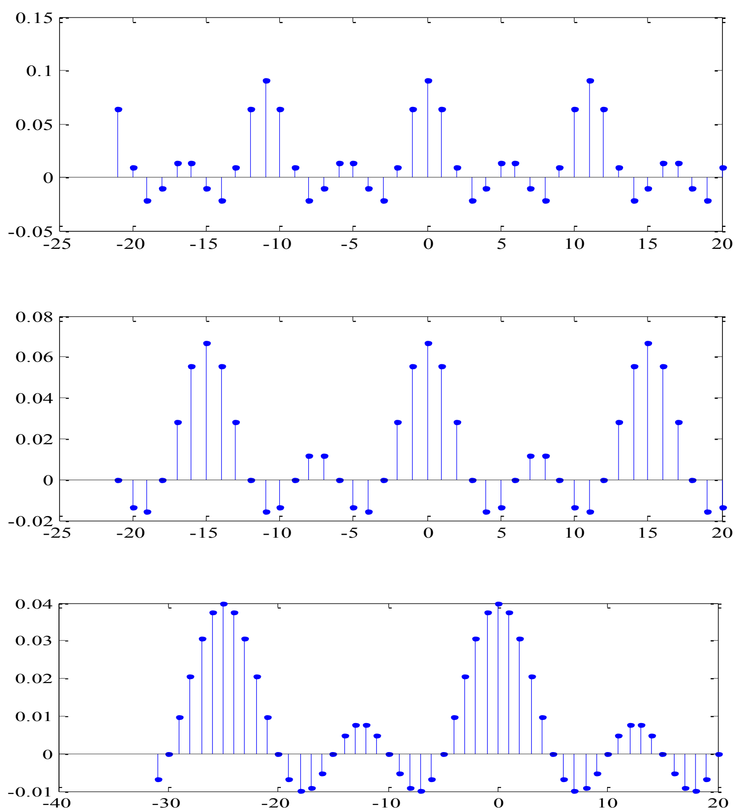

Figure 16 – Coefficients of the periodic pulses, (a) with N= 3, K0 = 7, (b) N =3, K0 = 15, (c) N = 3, K0 = 2

Summary

1. A discrete signal can be created by sampling a continuous signal with an impulse train of desired sampling frequency.
2. The sampling frequency should be greater the two times the highest frequency in the signal of interest.
3. The fundamental period of a discrete signal, given by K_0 must be an integer for the signal to be periodic.
4. The fundamental discrete frequency of the signal, given by Ω_0 is equal to $2\pi / K_0$.
5. The period of a digital frequency is an integer multiple of 2π. Harmonic discrete frequencies vary by integer multiple of 2π, such that Ω and $\Omega + 2\pi k$ are harmonic and identical.
6. Because discrete harmonic frequencies are identical, we cannot use them to represent a discrete signal.
7. Instead we divide the range from 0 to 2π by K_0 and use these digital frequencies as the basis set.
8. Hence there are only N = K_0 harmonics available to represent a discrete signal. The Fourier analysis is limited to these N harmonics.
9. Beyond the 2π range of harmonic frequencies, the discrete-time Fourier series coefficients, (DTFSC) repeat.
10. In contract, the continuous-time signal coefficients are aperiodic and do not repeat.
11. Sometimes we can solve the coeffcients using closed from solutions but in a majority of the cases, matrix methods are used to find the coefficients of a signal.
12. Matrix method is easy to setup but is computationally intensive.

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```
%Program Chapter 3 – Program 1

f0=1;
Fs = 32;
Ts1 = 1/Fs;
t = 0: Ts1: 2;
clf;
figure(1) % heavy sample
xt=cos(2*pi*t)- .3 + .6*sin(3*pi*t+.5)+.5*cos(4*pi*t)-
.3*cos(5*pi*t+.25);
ylabel('x[n]');
xlabel('Sample');
hold on
plot(t/Ts1,xt,'-. r' );
n = 0: 2*Fs;
xn1=cos(2*pi*n*Ts1)- .3 + .6*sin(3*pi*n*Ts1+.5)+.5*cos(4*pi*n*Ts1)-
.3*cos(5*pi*n*Ts1+.25);
stem(n, xn1);
hold off

figure(2) % light sample
plot(t/Ts1,xt,'-. r' );
hold on
n = 0: 2*4;
Ts1 = 1/4;
xn1=cos(2*pi*n*Ts1)- .3 + .6*sin(3*pi*n*Ts1+.5)+.5*cos(4*pi*n*Ts1)-
.3*cos(5*pi*n*Ts1+.25);
stem(n*8, xn1);
ylabel('x(t)');
xlabel('Sample');
hold off
```

```
%Chapter 3 - Program 2


t = 0: .01: 6;
x = .25*sin(2*pi*1*t)+.7*cos(2*pi*2*t)-
.5*cos(2*pi*3*t)+.15*sin(2*pi*4*t);
clf;
figure(1);
plot(t, x)
title('(a)')
ylabel('x(t)')
xlabel('Time, t')

figure(2);
n = 0: 47;
fs1 = 8;
xn8 = .25*sin(2*pi*1*n/fs1)+.7*cos(2*pi*2*n/fs1)-
.5*cos(2*pi*3*n/fs1)+.15*sin(2*pi*4*n/fs1);
plot(t*8,x, '--r')
ylabel('x[n]')
xlabel('Sample, n')
title('(b)')
hold on
stem(n, xn8, '.')
```

```
axis([ 0 48 -2 2]);
hold off

figure(3)
n2 = 0: fs2*6-1;
fs2 = 12;
xn12 = .25*sin(2*pi*1*n2/fs2)+.7*cos(2*pi*2*n2/fs2)-
.5*cos(2*pi*3*n2/fs2)+.15*sin(2*pi*4*n2/fs2);
plot(t*12, x , '--r')
ylabel('x[n]')
xlabel('Sample, n')
title('(c)')
hold on
stem(n2, xn12, '.')
axis([ 0 48 -2 2]);
hold off

figure(4);
n = 0: 47;
clf;
xnd = (1/48)*fft(xn8);
xnd2 = abs(fftshift(xnd));
plot(n, xnd2)
```

```
%Chapter 3 - Program 3

t = -.5: .01: .5;
y1 = cos(4*pi*t);
clf;

subplot(3,1,1)
plot(t, y1)
grid;
title('(a) ');
xlabel('Time, t seconds');
ylabel('x(t)');
axis;

subplot(3,1,2)
n = -5: 5;
y2 = cos(4*pi*n*.2);
stem(n, y2)

grid;
title('(b) ');
xlabel('Sample, n');
ylabel('x[n]');

subplot(3,1,3)
%n = -5: 5;
n2 = -2*pi: 2*pi/5: 2*pi;
y2 = cos(2*n2);
stem(n2, y2)
axis([-2*pi 2*pi -1 1]);
grid;
```

```
title('(c) ');

xlabel('Radians');
ylabel('x[n]');
axis([-2*pi 2*pi -1. 1.])
% Define x-ticks and their labels..
set(gca,'xTick',-2*pi: pi/5: 2*pi)
set(gca,'xTickLabel',{'-2pi', '', '-8pi/5', '', '', '', '-4pi/5', '',
'', '', '0', '', '', '', '4pi/5', '', '', '', '8pi/5' , '',  '2pi'})
```

```
%Chapter 3 - Program 4

f0=2;
Fs = 6;
t = 0: .001: 1;
n = 0: Fs*t;
n2 = 0: Fs

xt1=cos(2*f0*pi*t);
y = cos(2*f0*pi*n2/Fs)
xt2= cos(2*5*pi*f0*t);
figure(1)

plot(t*Fs, xt1, t*Fs, xt2, 'r')
hold on
stem(n2, y, 'filled')
```

```
%Chapter 3 - Program 5

f0=1;
Fs = 3;
t = 0: .001: 4;
n = 0: Fs*t;
n2 = 0: Fs*4

xt1=cos(2*f0*pi*t);
y = cos(2*f0*pi*n2/Fs)
figure(1)
grid;
plot(t*Fs, xt1, 'r')
xlabel('Sample, n')
ylabel('x[n]')
hold on

stem(n2, y, 'filled')
```

```
%Chapter 3 - Program 6


f0=.5/pi;
Fs = 1;
t = 0: .001: 30;
n2 = 0: Fs*30
```

```
xt1=cos(2*f0*pi*t);
y = cos(2*f0*pi*n2/Fs)
 figure(1)
 grid
plot(t*Fs, xt1, 'r')
xlabel('Sample, n')
ylabel('x[n]')
hold on

stem(n2, y, 'filled')
```

```
%Chapter 3 - Program 7
n = 0:40;
w = 2*pi/12;
phase = 0;
A = 1.0;
HShift = 2; %change this (even numbers only) to see effect of shift
x = A*cos((w+(HShift*pi))*n - phase);
clf;
stem(n,x, 'filled');    % Plot the generated sequence
axis([0 40 -1.25    1.25]);
grid;
title('Sinusoidal Sequence');
xlabel('Sample n');
ylabel('x[n]');
axis;
```

```
%Chapter 3 - Program 8

n=-12:12;
 N=9;
 w0=2*pi/N;

axis([-12.5 12.5 -1.1 1.1]);

k=0;
Phi0n=exp(j*w0*k*n);
subplot(3,4,1);
stem(n,real(Phi0n),'Marker','.');xlabel('1')
t= -10: 1/18: 10;
plot(t, cos(w0*k*t))
axis([-12.5 12.5 -1.1 1.1]);

k=1;
hold on
Phi1n=exp(j*w0*k*n);
subplot(3,4,2);stem(n,real(Phi1n),'Marker','.');xlabel('2')
t= -10: 1/18: 10;
plot(t, cos(w0*k*t))
axis([-12.5 12.5 -1.1 1.1]);
hold off
k=2;
Phi2n=exp(j*w0*k*n);
subplot(3,4,3);stem(n,real(Phi2n),'Marker','.');xlabel('3')
axis([-12.5 12.5 -1.1 1.1]);
```

```
k=3;
Phi3n=exp(j*w0*k*n);
subplot(3,4,4);stem(n,real(Phi3n),'Marker','.');xlabel('4')
axis([-12.5 12.5 -1.1 1.1]);
k=4;
Phi4n=exp(j*w0*k*n);
subplot(3,4,5);stem(n,real(Phi4n),'Marker','.');xlabel('5')
axis([-12.5 12.5 -1.1 1.1]);
k=5;
Phi5n=exp(j*w0*k*n);
subplot(3,4,6);stem(n,real(Phi5n),'Marker','.');xlabel('6')
axis([-12.5 12.5 -1.1 1.1]);
k=6;
Phi6n=exp(j*w0*k*n);
subplot(3,4,7);stem(n,real(Phi6n),'Marker','.');xlabel('7')
axis([-12.5 12.5 -1.1 1.1]);
k=7;
Phi7n=exp(j*w0*k*n);
subplot(3,4,8);stem(n,real(Phi7n),'Marker','.');xlabel('8')
axis([-12.5 12.5 -1.1 1.1]);
k=8;
Phi8n=exp(j*w0*k*n);
subplot(3,4,9);stem(n,real(Phi8n),'Marker','.');xlabel('9')
axis([-12.5 12.5 -1.1 1.1]);
k=9;
Phi9n=exp(j*w0*k*n);
subplot(3,4,10);stem(n,real(Phi9n),'Marker','.');xlabel('10')
axis([-12.5 12.5 -1.1 1.1]);
k=10;
Phi10n=exp(j*w0*k*n);
subplot(3,4,11);stem(n,real(Phi10n),'Marker','.');xlabel('11')
axis([-12.5 12.5 -1.1 1.1]);
k=11;
Phi11n=exp(j*w0*k*n);
subplot(3,4,12);stem(n,real(Phi11n),'Marker','.');xlabel('12')
axis([-12.5 12.5 -1.1 1.1]);
```

```
% Chapter 3 - Program 9
nmin = -10;
nmax = 9;
ND = abs(nmin)+nmax+1;

n = nmin: nmax;
x1 = 0.5 +  0.25*cos(2*pi*n/5) -  0.6*sin(2*pi*n/4);
clf
subplot(2,1,1)
stem(n,x1, '.');
pt = sum(x1.^2)*1/20
x1
title('a')
ylabel('x[n]')
xlabel('Sample n')

xnd = (1/ND)*dft(x1, ND);
subplot(2,1,2)
```

```
xnd2 = fftshift(xnd);
stem(n, abs(xnd2))
ylabel('X(f)')
xlabel('Harmonic k')
pf =  sum(abs((xnd2.^2)))
Title('b')
```

```
%Chapter 3 - Program 10
%Figure 12
a0 = [ 0 0 0 0]
d = [ .2 .7 1.1 .9 .5]
xom = [ a0 a0 a0 a0 a0 a0 a0 d a0 a0  a0  a0 a0 a0  a0  ]

n = 0: length(xom)-1;
N = 256;
figure(1)
stem(n, xom)
title('(d)')
figure(2)
X = fft(xom, N);
plot(abs(fftshift(X)))
w = 6*pi * (0:(N-1)) / N;
w2 = fftshift(w);
plot(w2)
w3 = unwrap(w2 - 2*pi);
plot(w3)
plot(w3, abs(fftshift(X)))
xlabel('radians')
plot(w3/pi, abs(fftshift(X)))

xlabel('radians / \pi')
```

```
%Chapter 3, Problem 11
om = 2*pi/4;
W = exp(-j*om)
for n = 1:4
    for k = 1: 4
       m(n,k)  = W^((n-1)*(k-1))
    end

end
m
x = [ 2 1 0 0];
(1/4)*x*m
```

```
% Chapter 3 - Program 12
N = 5;
K0 = 11;
n = -21:20;
coff = (1/K0)*diric(n*2*pi/K0, N);
stem(n, coff, '.')
```