Intuitive Guide to Fourier Analysis

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Much of this book relies on math developed by important persons in the field over the last 200 years. When known or possible, the authors have given the credit due. We relied on many books and articles and consulted many articles on the internet and often many of these provided no name for credits. In this case, we are grateful to all who make the knowledge available free for all on the internet.

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5 Discrete-time Fourier transform (DTFT) of aperiodic and periodic signals

Paul Adrien Maurice Dirac
8 August 1902 – 20 October 1984

Paul Adrien Maurice Dirac was an English theoretical physicist who made fundamental contributions to the early development of both quantum mechanics and quantum electrodynamics. He was the Professor of Mathematics at the University of Cambridge, and spent the last decade of his life at Florida State University. Among his discoveries, he formulated the Dirac equation, which describes the behavior of fermions and predicted the existence of antimatter. Dirac shared the Nobel Prize in Physics for 1933 with Erwin Schrödinger. He also did work that forms the basis of modern attempts to reconcile general relativity with quantum mechanics. Paul Dirac in his influential 1930 book The Principles of Quantum Mechanics. introduced the "delta function" which he used as a continuous analogue of the discrete Kronecker delta. – From Wikipedia

Whether periodic or non-periodic, discrete-time signals are the main-stay of signal processing. Signals are collected and processed via sampling, or by devices which are inherently discrete. Despite the fact that sampled signals "look" like their analog parents, there are

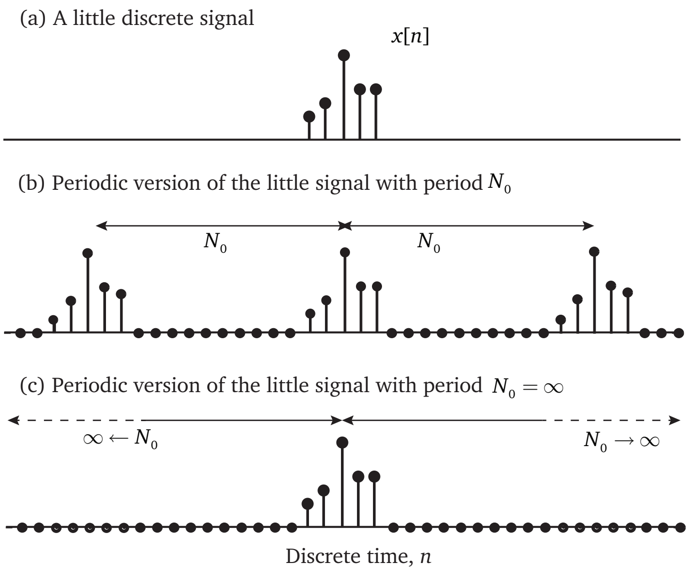

Figure 5.1: An aperiodic discrete-time signal can be considered periodic if period is assumed to be infinitely long.

some major conceptual differences between discrete and continuous signals. The fundamental difference of course between the CT and the DT signals is the frequency ambiguity that we experience for DT signals.

Let's do the same thought experiment we did for continuous signals. Given a piece of a discrete and ostensibly aperiodic signal such as in Fig. 5.1(a), we conceptually extend its period. This signal $x[n]$ is just 5 samples, but we can pretend that the signal is periodic with period N_0, with $N_0 >> 5$. But then we can also say that this period is very long, maybe even infinitely long. So if we extend the period of this signal to ∞, we basically get back the original signal, $x[n]$ of 5 samples, which is now surrounded by a sea of zeros.

$$\lim_{N_0 \to \infty} x_{N_0}[n] = x[n] \tag{5.1}$$

As we increase N_0, in limit the result is the starting signal, but it can now be considered a periodic signal, although only in a mathematical sense. We can't see any of the periods. They are too far apart. And now since the signal is periodic, we can use the discrete-time Fourier series (DTFS) to write its frequency representation in terms of its complex coefficients as

$$C_k = \lim_{N_0 \to \infty} \frac{1}{N_0} \sum_{n=0}^{N_0-1} x_{N_0}[n] e^{jk\Omega_0 n}. \tag{5.2}$$

Discrete-time Fourier Transform (DTFT)

In Chapter 4 we discussed the result of extending the period to infinity, which leads to a continuous frequency response, even though the signal itself is discrete. We do the same here for discrete signals.

In Chapter 3 we defined the fundamental digital frequency of a discrete periodic signal as $\Omega_0 = 2\pi/N_0$, with N_0 as period of the signal in samples. As N_0 goes to infinity, the fundamental frequency goes to zeros as well. We can think of the fundamental frequency as the resolution of the spectrum, so if this number is zero, then the frequency becomes continuous and k, the harmonic identifier drops out entirely. Hence there are no unique harmonics. Now the signal of Eq. (5.1) can be written as a periodic signal $x[n]$.

Now we define a new transform called the **Discrete-time Fourier Transform** (DTFT) for a discrete aperiodic signal, assuming that $N_0 = \infty$ as

$$\boxed{\text{DTFT } X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}} \tag{5.3}$$

Here $x[n]$ is an aperiodic discrete-time signal. The expression does not contain any reference to the harmonic index, k. Compare this to the CTFT as given by

$$\boxed{\text{CTFT } X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt} \tag{5.4}$$

The CTFT frequency is termed ω whereas the digital frequency for discrete signals is given by Ω. Notice the similarity between these two transforms. The CTFT, $X(\omega)$ is continuous in frequency. The DTFT or $X(\Omega)$ is also continuous in frequency for the same reason: due to the extension of the period to ∞. Both $X(\omega)$ and $X(\Omega)$ are continuous functions, hence we have written them with round brackets.

The inverse DTFT is similarly given by this expression.

$$\boxed{\text{iDTFT} \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega} \tag{5.5}$$

The forward transform or the DTFT is denoted by symbol $X(\Omega)$. However, you will find other ways of denoting the DTFT. Oppenheimer book [**?**] refers to it as $X(e^{j\Omega})$, whereas both

Miral book [?] and the Lathi [?] and Green's [?] books refers to it by $X(\Omega)$. These notations are basically convention, nothing to lose sleep over. In speaking, most all of these forms are referred to simply as the "Fourier Transform" or even the more generic "spectrum". And even more egregiously, often just called FFT, which it may or may not be correct. Since most signals we deal with in practice are discrete, the time qualifier can be dropped and we can just call it the Fourier transform. However, in this book we will continue to refer to each type by its full formal name, CTFT, DTFT, DFT etc.

DTFT is continuous and periodic with period of 2π

So first we say that the period of a signal is assumed to be infinitely long and now we are saying that the DTFT is periodic. How can that be? Yes, in time domain, we are assuming that the period is infinitely long. But in frequency domain the spectrum is periodic and $X(\Omega)$ repeats with 2π. This talk of a frequency that is measured in multiples of π can be confusing. But we must accept the fact that the DTFT is defined in terms of the *digital frequency* which is special type of frequency. The signal consists of discrete values and in order to make the analysis independent of real physical time, i.e. the time between the samples, the DTFT is defined in terms of the digital frequency. This, if you trust us, also makes the math, easier (ha!).

Unlike continuous frequency, the whole range of digital frequency is limited to 2π. The spectrum computed thereof is also limited to that range and is called the *principal alias* as we noted in Chapter 3. Because of this condition, the coefficients for harmonic frequencies outside 0 to 2π are just copies. Hence there is no need to compute $X(\Omega)$ outside this range. Anything beyond that just repeats the same values from the 2π range, or in fact from any such range. We can ignore all these "replicated spectrum" as they are identical to the principal alias. We write this property as

$$X(\Omega) = X(\Omega + 2m\pi) \qquad \text{for all } \Omega \in [-\pi, \pi],\ m \text{ an integer} \tag{5.6}$$

This comes from the observation that

$$\begin{aligned} X(\Omega + 2m\pi) &= \sum_{n=-\infty}^{\infty} x[n]e^{j(\Omega+2\pi m)n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{j\Omega n}\underbrace{e^{j2\pi mn}}_{=1} \\ &= X(\Omega) \end{aligned}$$

Every $2\pi, X(\Omega)$ is identical to the one before. This property simplifies the computation as we need only integrate over a 2π range of the digital frequency. Since the area under a periodic signal for one period does not change no matter where you start the integration, we can generalize the DTFT equation over any range. We can for example write the equation for the iDTFT in the second manner, with integration range written as just 2π, and both are valid.

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega
\end{aligned}
$$

Comparing DTFT with CTFT

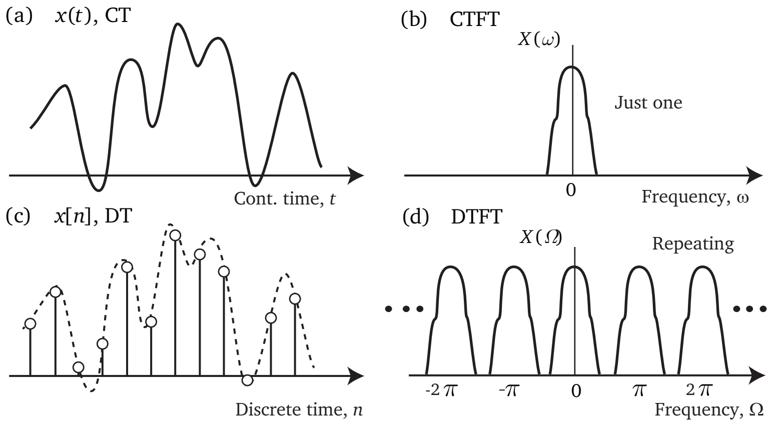

Figure 5.2: Comparing DTFT with CTFT (a) aperiodic CT signal, (b) its CTFT is continuous, (c) a sampled discrete signal (d) is same as (b) but repeats with 2π.

Both CTFT and DTFT have a very similar construct. Let's examine how DTFT differs from the CTFT for an *aperiodic* signal. Assume that the CTFT of the signal shown in Fig. 5.2(a) is as shown in (b). We see a single spectral mass around the zero frequency with a continuous frequency resolution. Now take the same signal in discrete form as in (c). This has a DTFT that is continuous just as the CTFT, but this one repeats with 2π radians. This is the same result we showed for discrete-time Fourier series. Both pairs of transforms, the CTFSC - CTFT, the DTFSC - DTFT are similar. And this strange result is the mathematical response to the frequency ambiguity of discrete signals. There are all served to use by the DTFT for our consideration.

We will now show some examples of the DTFT. In these examples, we compute only the principal alias which is the DTFT around the zero frequency, from $-\pi$ to π. However, we must not lose sight of the fact that the DTFT spectrum copies go on forever on each side of the principal alias, as we see conceptually in Fig. 5.2(d).

The CTFT properties shown in Chapter 4, Table 4.1 are also valid in conceptual sense for discrete signals. These properties can be used to compute the DTFT for many signals. We can, in most cases, take a CTFT equation, change the continuous frequency ω to digital frequency notation Ω and then change continuous time t to discrete time notation n and get a valid expression for the DTFT. However, what we get this way is only the principal alias because CTFT does not repeat.

The DTFT is a bridge topic to get us to the **Discrete Fourier transform** (DFT), a widely employed and a very useful algorithm. DFT is discrete in both time and frequency domain and can be calculated easily by software such as Matlab. The **Fast Fourier Transform** (FFT) was developed to make computation of the DFT quick and efficient. FFT is of course just an algorithm for computing the DFT efficiently and not a unique type of Fourier transform on its own.

The DTFTs for most signals other than a few simple ones you see in text books are hard to compute, requiring one to pull out integral tables. Nor are they commonly used in real-life engineering. So why bother with the DTFT if the subject is so theoretical? The main reason is that until we understand the DTFT, we cannot fully appreciate the DFT. When we learn it as a stand-alone topic, the DFT makes sense only in a procedural sense but one lacks deeper understanding of where it is coming from. Since this book is all about deep understanding, we ask you to read this chapter carefully.

Obtaining a DTFT from CTFT

DTFT can be obtained directly from a CTFT. Let's compute the DTFT of a signal whose CTFT we know. Take a signal that is a constant of magnitude 1. Its CTFT is an impulse of magnitude 2π. (See Example 4.1). What if we have a constant of magnitude 1 in discrete-time domain, what is its DTFT?

$$
\begin{aligned}
x(t) &= 1 \longleftrightarrow X(\omega) = 2\pi\delta(\omega) \\
x[n] &= 1 \longleftrightarrow X(\Omega) = ?
\end{aligned}
$$

This is a trivial case. By making the appropriate changes, we get

$$X(\Omega) = 2\pi\delta(\Omega), \qquad \text{for } -\pi \leq \Omega \leq \pi.$$

We get the same impulse at the origin as for a CTFT, of magnitude 2π but, we then also get its copies at all integer multiples of the 2π, the range of the digital frequency. The complete DTFT repeats, so we extend the above expression to

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \qquad \text{for all } \Omega.$$

For each k, we get an impulse at frequency $\Omega = 2\pi k$, so this says that the spectrum of a constant discrete signal is ever repeating impulses at integer multiples of 2π.

DTFT of a delayed impulse

The delayed impulse $x[n] = \delta[n - n_0]$ is a very important signal. Nearly all discrete signals can be represented as a summation of this general signal. We can compute the DTFT by taking the CTFT of the delayed impulse and change the terms to their discrete equivalents but, instead we will do the actual math using the DTFT equation. We compute the DTFT of a delayed unit-impulse function, $x[n] = \delta[n - n_0]$ using the DTFT Eq. (**??**).

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\Omega n}$$

The product of functions, $\delta[n - n_0]$ and $e^{j\Omega n}$ is non-zero only at point n_0, so we simplify the RHS as

$$X(\Omega) = e^{-j\Omega n_0}$$

Hence the DTFT of a discrete delayed delta function is a CE of frequency n_0.

Therefore

$$x[n] = \delta[n - n_0] \longleftrightarrow X(\Omega) = e^{-j\Omega n_0} \tag{5.7}$$

The magnitude and the phase of this transform is equal to

$$|X(\Omega)| = |e^{-j\Omega n_0}| = 1$$

$$\angle X(\Omega) = \arctan \frac{\sin(\Omega n_0)}{\cos(\Omega n_0)}$$

So no matter what the shift, the magnitude remains the same. The phase however will change with the shift. This result is exactly the same as if we had applied the time-shift property to a zero-shift delta function. If the shift is equal to 0, then we get

$$x[n] = \delta[n - 0] \longleftrightarrow X(\Omega) = e^{-j\Omega(n_0 = 0)} = 1 \tag{5.8}$$

The transform of the un-shifted delta signal is of course 1 as we see in Figure **??**(b) and we can see that the DTFT of this signal is a purely continuous function of Ω. If $n_0 = 2$, we get

$$x[n] = \delta[n-2] \longleftrightarrow X(\Omega) = e^{-j2\Omega} = \cos(2\Omega) - j\sin(2\Omega)$$

In figure 5.3, we see the effect of the delay on the transform of the delayed function, with no change in magnitude but the phase change by 4π with 2π phase delay per sample delay. We see a total of 4π phase travel over the range in (f).

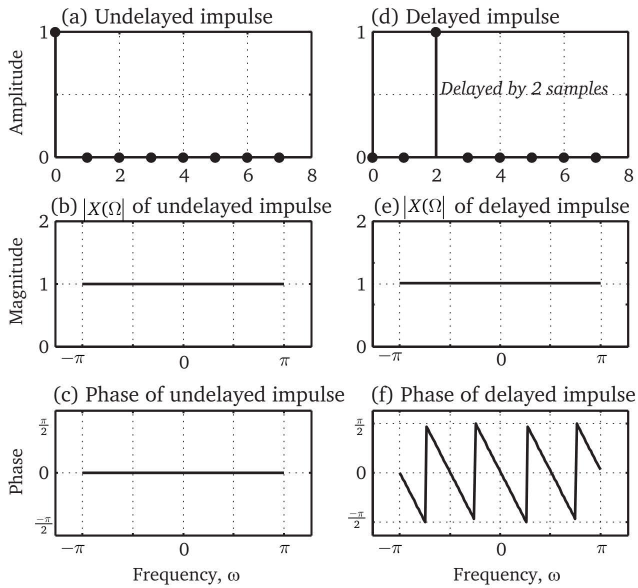

Figure 5.3: Comparing $X(\Omega)$ of an unshifted and shifted impulse.

DTFT of superposition of impulses

Example 5.1. We are going to examine the DTFT of a group of pulses, first 3 and then 5, all centered at zero, as shown in Fig. 5.4. We can compute the DTFT by treating each impulse as an independent signal and hence its DTFT is equal to a CE of a frequency equal to the its delay term. We compute the DTFT of the 3 impulse function very simply as

$$\begin{aligned} x[n] &= \ldots 111000\ldots \\ &= \delta[n+1] + \delta[n] + \delta[n-1] \\ X(\Omega) &= e^{j1\Omega} + 1 + e^{-j1\Omega} \end{aligned}$$

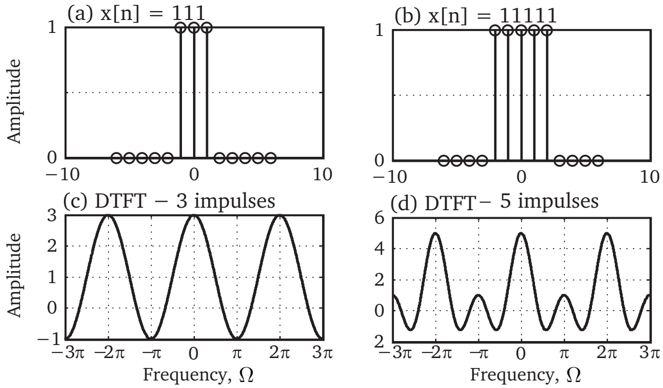

Figure 5.4: DTFT of a group of pulses. (a) a 3-impulse signal and (b) a 5-impulse signal.

Similarly for the 5-impulse case, we can write it as

$$
\begin{aligned}
x[n] &= ...111110... \\
&= \delta[n+2]+\delta[n+1]+\delta[n]+\delta[n-1]+\delta[n-2] \\
X(\Omega) &= e^{j2\Omega}+e^{j1\Omega}+1+e^{-j1\Omega}+e^{-j2\Omega}
\end{aligned}
$$

Note that the DTFT of the 3-impulse signal can be analyzed easily. The center impulse results in a DTFT of 1.0. The two adjacent impulses represent a cosine of frequency 1, hence the DTFT should be the cosine wave with a DC offset of 1.0, which is what it is. We can think out the DTFT of the 5 impulses as well. It is the sum of a DC offset of 1.0, plus a cosine of frequency 1 and another one of frequency 2.

Example 5.2. Compute the DTFT of a discrete signal that combines several shifted impulse functions.

$$x[n] = \delta[n]+2\delta[n-1]+4\delta[n-2]$$

We treat each one of these delta functions individually by applying the linearity principal.

$$
\begin{aligned}
X(\Omega) &= \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega n} + 2\sum_{n=-\infty}^{\infty} \delta[n-1]e^{-j\Omega n} + 4\sum_{n=-\infty}^{\infty} \delta[n-2]e^{-j\Omega n} \\
&= 1 + 2e^{-j\Omega} + 4e^{-j2\Omega} \\
&= 1 + 2\cos(\Omega) - j2\sin(\Omega) + 4\cos(2\Omega) - j4\sin(2\Omega) \\
&= \underbrace{1 + 2\cos(\Omega) + 4\cos(2\Omega)}_{\text{real}} - j\underbrace{(2\sin(\Omega) - 4\sin(2\Omega))}_{\text{imaginary}}
\end{aligned}
$$

Note that since the digital frequency Ω has units of radians, we do not have a time variable to go along with it. We note that the DTFT is continuous and repeats with 2π. The spectrum shown covers 3 periods, the rest are all there, outside the boundaries of the plot. We don't show them but they are indeed there, all looking like the principal alias in (b).

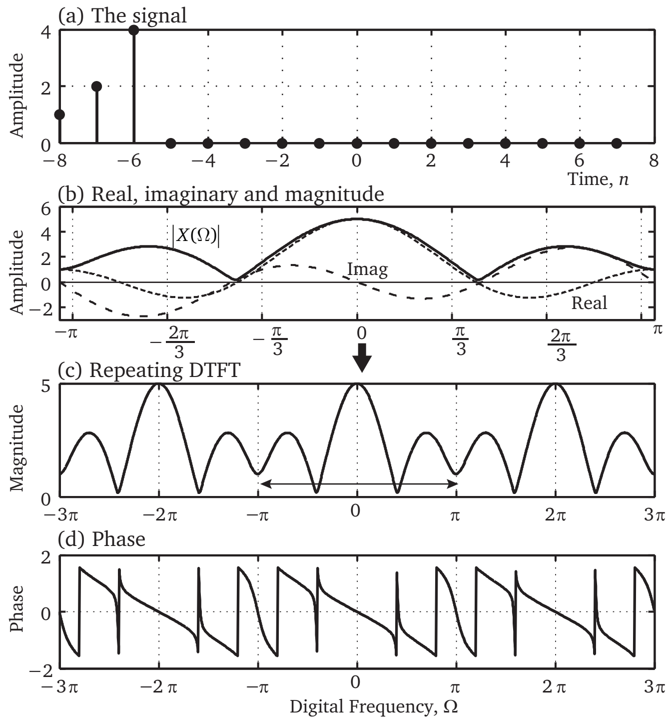

Figure 5.5: DTFT of (a) the discrete aperiodic signal, (b) Principal alias, (c) the real repeating DTFT over three 2π sections (d) the phase.

DTFT of a square pulse

Square waves are very useful as a model of real signals. We have been examining them in each chapter. What you learn from these signals, you can then generalize to nearly all shapes. Let's start with a square pulse of width N discrete samples. However, note that signal period is not N, it is longer and in fact is it not infinitely long? Hence the parameter N has nothing to do with the length/period of the signal. It is just the width of the pulse itself. Note that in this example, the square pulse is centered at 0. We assume that N is odd. We define this function as

$$x[n] = \begin{cases} 1 & -M < n \leq M, \text{where } M = (N-1)/2 \\ \text{elsewhere} & \end{cases}$$

We compute the DTFT as

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \\ &= \sum_{n=-M}^{M} 1 \cdot e^{-j\Omega n} \\ &= \frac{\sin\left(\frac{2M+1}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)} \end{aligned} \tag{5.9}$$

The result in the last row is the **Dirichlet Function**. We can also write the result as follows

$$X(\Omega) = \text{Diric}(N, \Omega) \tag{5.10}$$

We plot the DTFT using Eq. (5.10) for various values of N, which is the width of the square pulse in samples. The absolute value of the Dirichlet function is plotted vs. the true value in the RHS of Fig. 5.6. The length of the signal is 12 samples for each case. Can you say what would happen to the DTFT (on the RHS), if we increase the length of the signal from 12 samples to 100 samples. Actually nothing would change, we would get exactly the same result. DTFT is not a function of the number of samples beyond the pulse. The DTFT as we can see in Eq. (5.10) is a function of only the number of samples of the square pulse or N. That's because the formulation of the DTFT already assumes that zeros on the sides go on forever.

Note that the result we got for Example 1 which are of course square pulses, is identical to the one computed using the diric function in Fig. The only difference is that one is plotting the absolute values so lobes appear flipped up.

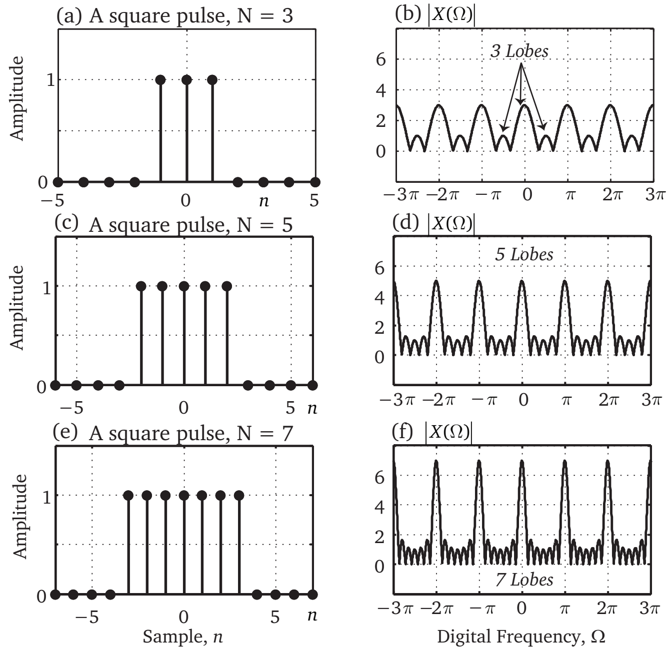

Figure 5.6: A pulse of length N = 3, 5, 7 samples and its spectrum.

Dirichlet detour

The Dirichlet function is often called the periodic version of the *sinc function*. These two are phenomenally important functions in signal processing.

$$\mathrm{sinc}(\omega) = \frac{\sin(\omega)}{\omega}$$

$$\mathrm{Diric}(\Omega) = \frac{\sin\frac{N\Omega}{2}}{N\sin\frac{\Omega}{2}}$$

Notice the presence of N in the diric function. Diric is a periodic function and hence N represents the period. There is no such parameter in sinc since it is not a periodic function. So the two functions are not the same or even equivalent but are often presumed to be so. We plot both of these functions in Fig. 5.7 to see the relationship. The sinc function in the top row is continuous and *aperiodic*. In Matlab, we use the normalized version of the sinc function such that the x-axis is given in terms of x. Only when x is an integer is the function zero. So in a discrete sampling, the sinc function is equivalent to a single impulse.

The Dirichlet in the second row has the parameter N, the width of the square pulse in samples. It is periodic with 2π for $N =$ odd, and 4π when N is even. (We don't see this in Fig. 5.7, because the plot contains absolute values so all the lobes are on the positive side.)

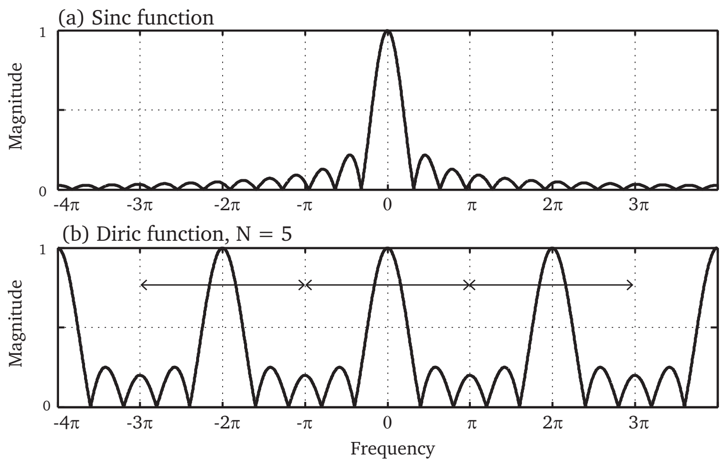

Figure 5.7: Sinc and the Dirichlet function

Let's examine the Dirichlet function in a bit more detail. Fig. 5.8 shows the behavior of the the Diric function (the Matlab version of the Dirichlet). We see that the number of zero crossings in the range of 2π are equal to $N-1$. For $N=5$, we see 4 zero crossings, for $N=9$, we see 8 zero crossing. The function is non-zero only at 0, exactly the same as a sinc function in the 2π range. On RHS, we see the same function over a longer range of digital frequencies plotted along the sampled-discrete version.

Where the sinc function looks like a single impulse when sampled at integer arguments, the sampled Dirichlet looks like an *impulse train*, with impulses present every 2π. The Diric function crosses zeros at all frequencies equal to $(2\pi m)/N$ where N is the order of the diric function as in Matlab `diric(f, N)`. Hence for $N = 5$, the zeros occur at $\Omega = \pm 2\pi/5,\ \pm 4\pi/5, \pm 6\pi/5,\ \pm 8\pi/5, \ldots$, for N = 6, the zeros occur at $\Omega = \pm 2\pi/6,\ \pm 4\pi/6,$ $\pm 6\pi/6,\ \pm 8\pi/6, \ldots$ etc. The parameter N determines the number of lobes before the function starts repeating again.

Applying time-shift property to the DTFT of a square pulse

What is the DTFT of a square pulse, when not centered at 0? We can think of this as a square pulse located at zero frequency but with a time-shift. Knowing the time-shift property is a very handy thing. The analysis is same as in un-shifted case, except we are going to add a time shift. We assume that the pulses are centered at L samples from the origin. The time-shift is L units. The DTFT can now be written from the time shift property as simply the

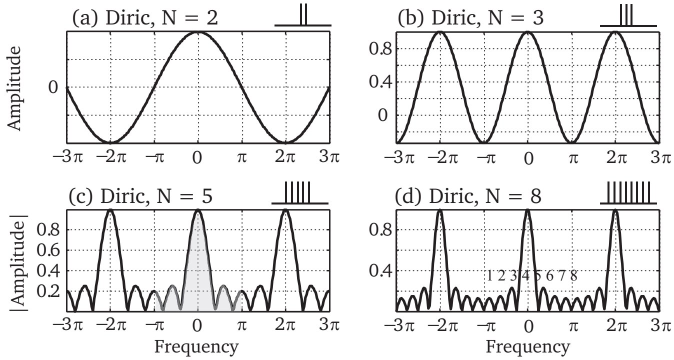

Figure 5.8: The Dirichlet function (a) $M = 0, N = 2$, (b) $M = 1, N = 3$, (c) $M = 2, N = 5$, (d) $M = 3, N = 8$. The x-axis is in radians. The zero crossings occur at $2\pi n/N$. Note that there is only one peak in each 2π range for all cases and the number of lobes in this range is equal to N. Case (a) is of course a sinusoid because the signal for $N = 2$ is just two impulses.

DTFT times the CE of frequency per Eq. (5.7) $e^{-j\Omega L}$ as follows

$$\begin{aligned} X(\Omega) &= e^{-j\Omega L} X(\Omega)_{\text{undelayed}} \\ &= e^{-j\Omega L} \frac{\sin\left(\frac{2N+1}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)} \end{aligned} \tag{5.11}$$

In Fig. 5.9, we plot the DTFT for two pulse widths, $N = 3$, with a time shift of $L = 10$ samples. On the LHS, we see the un-shifted square pulse, on the RHS, the shifted version. We see from Eq. (5.11) that the magnitude of the DTFT did not change, the shift results in no change in the magnitude of the DTFT, only the phase.

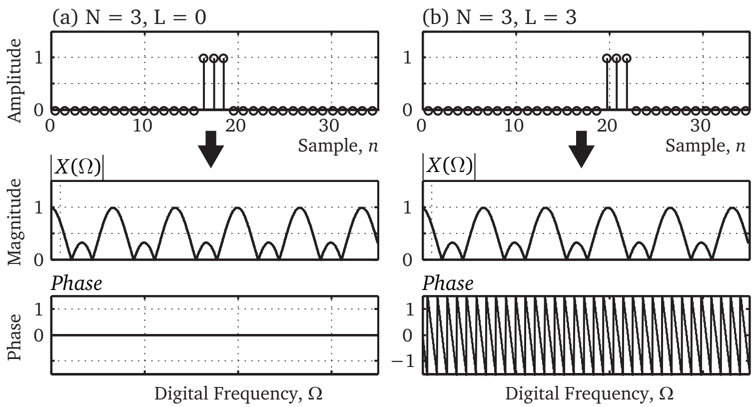

Figure 5.9: Time shift property for $N = 3$.

Time expansion property

Example 5.3. Let's take this signal which looks like it has zeros inserted in between the samples of a 3 sample square pulse, $x = [\,1\,0\,0\,1\,0\,0\,1\,]$. We can write this discrete signal as

$$x[n] = \delta[n] + \delta[n-3] + \delta[n-6]$$

There are many different ways of computing the DTFT of such a signal. We can do it such as we did with the individual components in Example (). Here we apply the time-expansion property to show a more efficient method. The pulse of $N = 3$ has been expanded by a factor of 3 by inserting these zeros. We write the time-expansion property as

$$x(at) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\Omega}{a}\right) \tag{5.12}$$

We see that adding zeros between the samples expands the signal but compresses the DTFT. In Fig. 5.10, we see this effect as more zeros are added. Why exactly does that happen? Examine the case of the 3-impulse square pulse. As we increase the spacing between, the outer two pulses by the addition of zeros, we are increasing the frequency of the real signal represented by these two outer impulses. What happens if N goes to infinity? Then only the center delta function is left, and the DTFT will turn into a flat line, as we can guess from the compression being seen in Fig. 5.8.

DTFT of a triangular pulse

Example 5.4. A triangular pulse is nearly as important in signal processing as the square pulse. It is the convolution of two rectangular pulses, something which comes up often. We write the triangular pulse as

$$x[n] = 1 - \frac{|n|}{N}, \qquad |n| < N$$

The pulse is 2N samples wide and symmetrical. The DTFT is computed as

$$\begin{aligned}
X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n] e^{j\Omega n} \\
&= 1 + \sum_{n=1}^{N-1} \left(1 - \frac{n}{N}\right)\left(e^{j\Omega n} + e^{-j\Omega n}\right) \\
&= 1 + 2\sum_{n=1}^{N-1} \left(1 - \frac{n}{N}\right) \cos(n\Omega) \\
&= \frac{\sin^2\left(\frac{N}{2}\Omega\right)}{N \sin^2\left(\frac{1}{2}\Omega\right)}
\end{aligned} \tag{5.13}$$

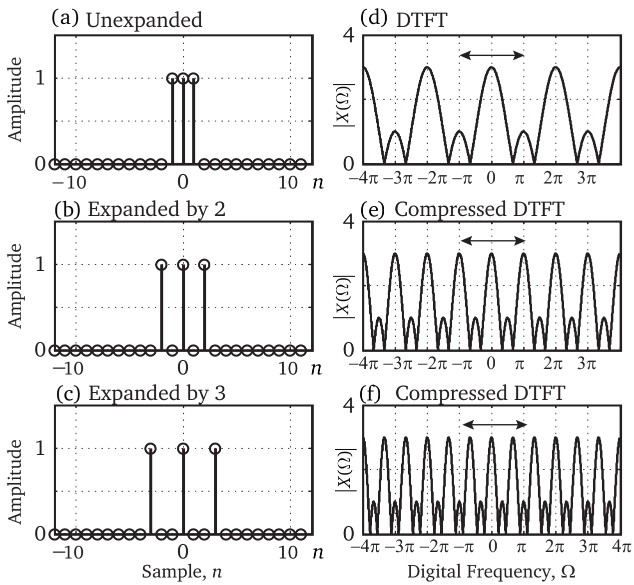

Figure 5.10: Time expansion property. As the impulses are spreading out, their effect is lessening and hence what remains is just the center pulse in a limiting case. We see the response compressing and approaching a single continuous line with increasing a, the spreading factor.

The result is a function that is the Dirichlet function squared.

We could have computed the DTFT of a triangular function by applying the convolution property. We recognize, that a triangle pulse is the result of a convolution of two identical rectangles. So we write the pulse as a convolution.

$$x[n] = \text{rect}\left[\frac{n}{N}\right] * \text{rect}\left[\frac{n}{N}\right] \tag{5.14}$$

The DTFT of this convolution is the product of the DTFT of the individual square pulses. From that we get

$$X(\Omega) = \left[\frac{\sin\left(\frac{2N+1}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}\right]^2 \tag{5.15}$$

So knowing the properties can make the task of computing FTs easier in many cases.

Computing the DTFT of a Raised-cosine pulse

The sinc pulses are great and it would be wonderful if we could actually build them but they require an infinite length and can only be approximated. The alternate options are

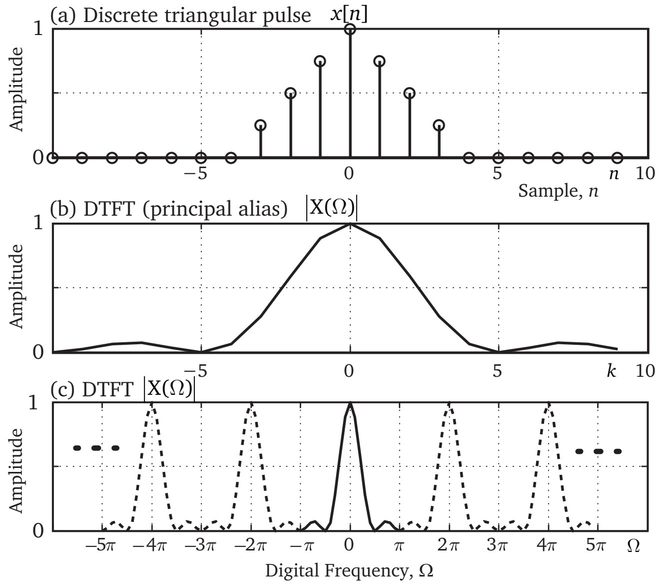

Figure 5.11: A triangular shape pulse has a squared Diric signal response.

raised-cosine pulses which also limit the bandwidth of the baseband signal and are easily built in hardware.

$$p[n] = \frac{\cos\left(\pi\alpha\dfrac{n/F_s}{T_s}\right)}{1-2\alpha\dfrac{n/F_s}{T_s}} \times \frac{\sin\left(\pi\dfrac{n/F_s}{T_s}\right)}{\pi\dfrac{n/F_s}{T_s}} \tag{5.16}$$

Here F_s is the sampling frequency, α is a real number less than one and is called the roll-off factor, T_s is the inverse of symbol rate R_s. The first part is called the raised-cosine and the second part which is the sinc function is called the cascaded sinc applied to the raised-cosine pulse. If $\alpha = 0$, we get an ideal rectangular shape, and if $\alpha = 1$, we get a pure raised-cosine shape. These parameters set the baseband bandwidth of the signal as

$$\text{BW} = R_s(1+\alpha) \tag{5.17}$$

To compute the DTFT of this pulse we will have to resort to some heavy-duty math. But no need, as it has already been done for us by better minds. Here is the equation that gives us the CTFT of the above good looking pulse.

$$P(f)=\begin{cases} T_s & 0\leq |f| \leq \dfrac{1-\alpha}{2T_s} \\ \dfrac{T_s}{2}\left\{1+\cos\left[\dfrac{\pi T_s}{\alpha}\left(|f|-\dfrac{1-\alpha}{2T_s}\right)\right]\right\} & \dfrac{1-\alpha}{2T_s}\leq |f| \leq \dfrac{1+\alpha}{2T_s} \\ 0 & |f|\geq \dfrac{1+\alpha}{2T_s} \end{cases} \tag{5.18}$$

We plot the time-domain signal and its DTFT in Fig. 5.11. It looks very similar to a sinc function. Although this pulse too goes on forever, for practical design, it is clipped to a certain length, referred to by *taps*.

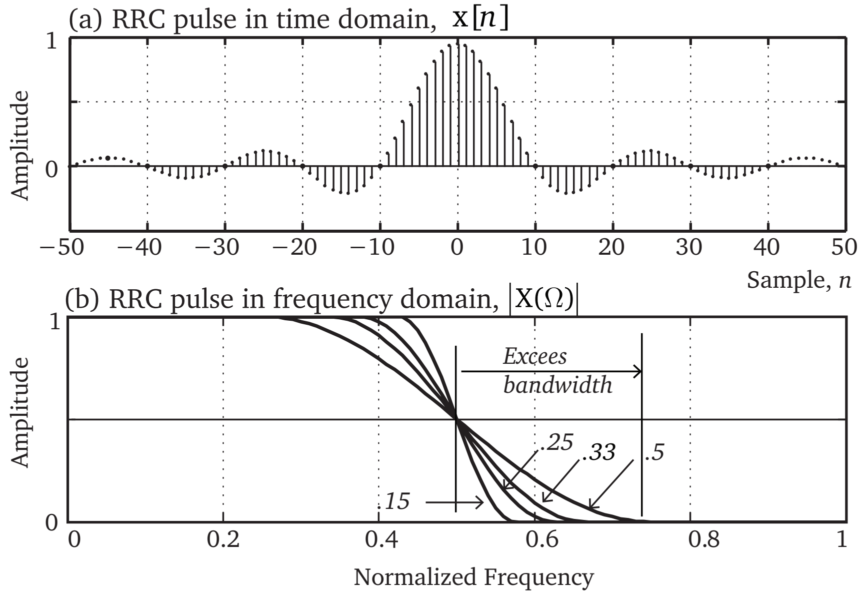

Figure 5.12: (a) Time domain root-raised cosine pulse shape (b) The spectrum of the raised cosine pulse for $\alpha = 0.5, 0.33, 0.25, 0.15$. Note that frequency domain looks like a low pass filter, with a roll-off.

DTFT of a Gaussian pulse

The discrete-time version of the Gaussian signal is given by

$$x[n]=\frac{1}{\sigma\sqrt{2\pi}}e^{-n^2/2\sigma^2} \tag{5.19}$$

To compute the DTFT of this signal, being good engineers we are going to again skip the math. It is a fairly simple calculation, but others have already done it for us. The DTFT of this Gaussian shaped pulse, is also Gaussian in shape. You recognize why that happens; because the pulse is an exponential and the integral of such a function is also an exponential. The result is beautiful and elegant and a very useful thing to know. Many random signals are Gaussian in nature.

$$X(\Omega) = e^{-\Omega^2/2\sigma^2} \tag{5.20}$$

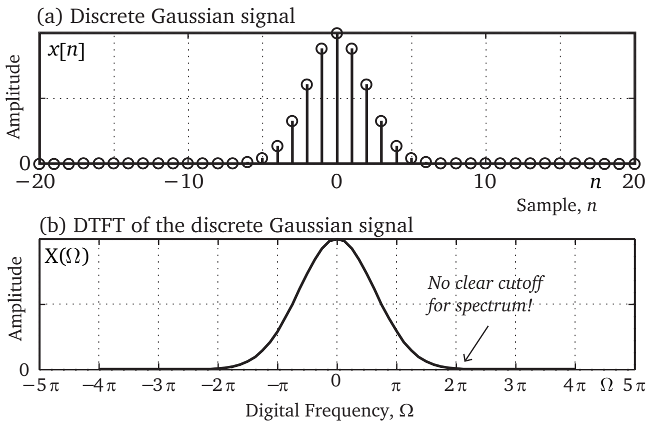

Figure 5.13: DTFT of a Gaussian pulse for $\sigma = 2$. Note that there is no obvious cutoff.

Note that we would have to sample the Gaussian function by at least 4 times the maximum frequency in order to avoid significant aliasing. The reason is that this function has no obvious maximum frequency and no matter what we choose, the signal will still contain frequencies higher than that number albeit in low amplitudes. The alternate is to pass the signal first through a low pass filter which removes all higher frequencies.

Now the DTFT of periodic signals

DTFT is a mathematical concept and requires integration, hence is not a practical algorithm for anything more complicated than simple functions. Continuous and infinitely long functions are manageable in textbooks but impractical in real life. So a *continuous frequency spectrum* is not a desired result. What we want is a *discrete* spectrum, which is far more practical. As engineers working with numbers, we want a spectrum that is discrete and one which we can compute in a discrete manner using computers. However, DTFT does not give us that.

All of the signals we looked at so far in this chapter were aperiodic, pulses standing alone. But what about discrete signals that are periodic? We have a transform for these as well and this is a yet one more type of Fourier transform. The DTFT of periodic signals is the most important type of Fourier Transform. Not because periodic signals are so important but because, the DTFT of periodic signals is discrete. This is our desired goal. We want a discrete spectrum! The DTFT of periodic signals, when modified for finite-length signals, gives the Discrete Fourier Transform (DFT), the most used form and for which the well-known Fast Fourier Transform algorithm was written. It took us a lot of pages in this book and 100's of years of history to get to this important point.

But what if our signal is not periodic, then what? Never fear, we will just go ahead and pretend that it is periodic, with signal length equal to the period.

However, we are not quite there yet. Let's take a periodic, discrete-time signal with a period of N_0 samples and write its discrete Fourier series equation. Note we did not talk about a period when discussing DTFT of aperiodic signals, but we will now. Period now becomes relevant because these signals are periodic, so they have a period! And whenever, we have a period, the frequency resolution must be discrete. However to derive a transform for periodic discrete signals, we have to go back to *discrete-time Fourier series* as our starting point, same as we did for the continuous-time CTFT for periodic signals in Chapter 4.

The Fourier series is written in form of Fourier series coefficients for discrete-time signals as follows. (See chapter 3)

$$x[n] = \frac{1}{N_0} \sum_{k=0}^{N_0-1} C_k e^{j\frac{2\pi}{N_0}kn} \tag{5.21}$$

Where $\Omega_0 = 2\pi/N_0$ is the digital frequency of the discrete signal and N_0 is the period of the signal in samples. The coefficients of the harmonics are given by

$$C_k = \sum_{n=0}^{N_0-1} x[n] e^{-j\frac{2\pi}{N_0}kn} \tag{5.22}$$

Since now we have N_0 samples of a periodic signal, we can indeed compute these coefficients. Let's take the DTFT of Eq. (5.21).

$$\begin{aligned} X(\Omega) &= \mathfrak{F}\left\{ \frac{1}{N_0} \sum_{k=0}^{N_0-1} C_k e^{j\frac{2\pi}{N_0}kn} \right\} \\ &= \underline{C_k\, \mathfrak{F}\left\{ \frac{1}{N_0} \sum_{k=0}^{N_0-1} e^{j\frac{2\pi}{N_0}kn} \right\}} \end{aligned} \tag{5.23}$$

[Check this equation].

The coefficients in Eq. (5.23) are not a function of frequency, so they are pulled out in front. The DTFT of the underlined part, a summation of complex exponentials is a train of impulses.

$$\mathfrak{F}\left\{\sum_{k=0}^{N_0-1} C_K e^{j\Omega_0 kn}\right\} = 2\pi C_k \sum_{m=-\infty}^{\infty} \delta(\Omega - k\Omega_0 - 2\pi m)$$

Substituting this expression into Eq.(5.23), we get the equation for the DTFT of a periodic signal.

$$\boxed{X(\Omega) = 2\pi C_k \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{N_0}\right)} \tag{5.24}$$

This equation says that the DTFT of a periodic discrete signal repeats the DTFS coefficients, C_k at every integer multiple of the digital frequency. That's what the second part, the impulse train is saying. This formulation is quite different from the DTFT of an aperiodic signal which we computed by Eq. (**??**). The situation here is analogous to the case of the CTFT of periodic signals. The CTFT of a periodic signal is a discrete version of the CTFSC. Similarly the DTFT of a periodic signal is also a discrete version of the DTFSC. The only difference, and a very big one, is that the coefficients of the DTFT are periodic. This of course has to do with the frequency ambiguity of discrete signals.

	Periodicity	
Transfrom	*Aperiodic*	*Periodic*
CTFT	Continuous Frequency Resolution Non-repeating Spectrum	Discrete Frequency Resolution Non-repeating Spectrum
DTFT	Discrete Frequency Resolution Spectrum repeats with 2π	Discrete Frequency Resolution Spectrum repeats with Sampling Frequency

Figure 5.14: Four versions of the Fourier transform.

The DTFT of periodic discrete-time signals in Eq. (5.24) tells us that the DTFT of a periodic signal consists of its DTFS coefficients repeated every N_0 samples. Since N_0 is a finite number, the samples are discrete and no longer continuous as they are for an aperiodic case. The spectrum is now *discrete*. Just what we like! Now we are getting somewhere.

Repeating this important fact again: The *DTFT of both the aperiodic and the periodic signal repeats but is discrete only for the periodic signals.*.

In Fig. 5.15 we see the comparison of the CTFT and DTFT of a periodic signal. The CTFT of a continuous-time periodic signal is discrete but non-repeating. The DTFT of a discrete signal is discrete, however, it repeats with the sampling frequency F_s samps or N_0 in samples.

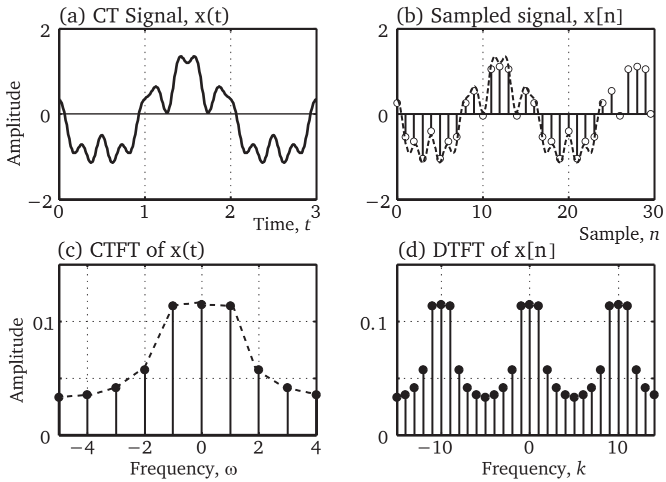

Figure 5.15: Comparing CTFT and DTFT for periodic signals.

Since the DTFT of a periodic signal is repeating the DTFS coefficients, here we give a table of the DTFS coefficients for some important signals. Knowledge of these makes computing the DTFT of periodic signals easier.

Now we look at a basic discrete-time signal that is periodic. As we shall see, the DTFT instead of being continuous is discrete.

DTFT of periodic signals

Example 5.5. We want to compute the DTFT of this periodic signal of period $N = 4$, $x[n] = [0, 1, 2, 1, 0, 1, 2, 1, ...]$.

In Fig. 5.16(a), we see the signal and its DTFT. For $N = 4$, we find that impulses occur in the frequency domain every $\pi/4$ radians. We have the same situation as for the CTFT. These coefficients are also a factor of 2π larger than the DTFSC.

Table 5.1: DTFS and DTFT of common function

Time Domain Signal $x[n]$	**DTFS** $C_k = \frac{1}{N_0}\sum_{n=-N_0/2}^{N_0/2} x[n]e^{-j\Omega_0 n}$	**DTFT** $X(\Omega) = \frac{1}{N_0}\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega_0 n}$
1	1	$2\pi\sum_{k=-\infty}^{\infty} C_k\delta\left(\Omega-\frac{2\pi k}{N_0}\right)$
$\delta[n]$ impulse at 0	does not exist	1
$\delta[n-n_0]$ shifted impulse	does not exist	$e^{-j\Omega n_0}$
$\sum_{m=-\infty}^{\infty}\delta[n-mN_0]$ Impulse Train, period N_0	$\frac{1}{N_0}$	$\frac{2\pi}{N_0}\sum_{k=-\infty}^{\infty}\delta\left(\Omega-\frac{2\pi k}{N_0}\right)$
$e^{jn\Omega_0}$ Periodic complex exponential with $\Omega_0 = \frac{2\pi}{N_0}$	$\begin{cases} 1 & k = mN_0 \\ 0 & \text{elsewhere} \end{cases}$	$2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)$
Cosine, periodic	$\begin{cases} 1/2 & k = mN_0 \\ 0 & \text{elsewhere} \end{cases}$	$\pi\sum_{k=-\infty}^{\infty}\delta(\Omega+\Omega_0-2\pi k)$ $+\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)$
Sine, periodic	$\begin{cases} j1/2 & k = mN_0 \\ 0 & \text{elsewhere} \end{cases}$	$j\pi\sum_{k=-\infty}^{\infty}\delta(\Omega+\Omega_0-2\pi k)$ $-j\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)$

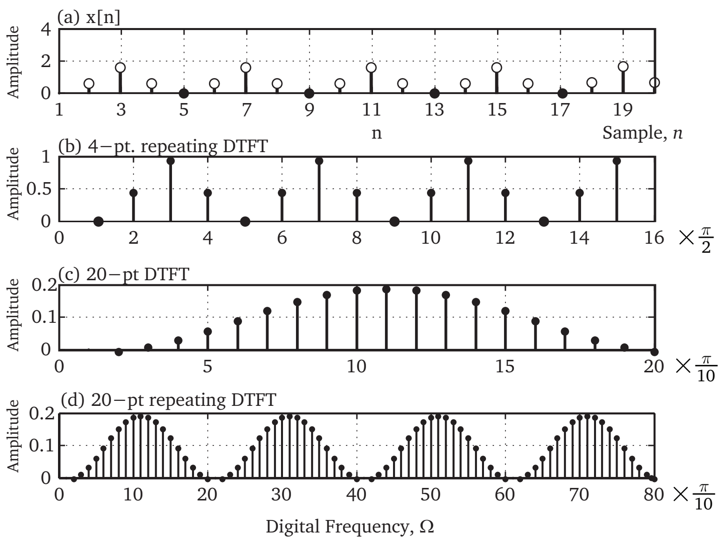

Figure 5.16: The DTFT of a periodic signal. Instead $N = 4$, we have plotted the spectrum for $N = 20$.

To compute the DTFT we note that the fundamental period is equal to 4. The DTFT of this signal is given by Eq. (5.24) by

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} C_k \delta\left(\Omega - \frac{2\pi k}{4}\right)$$

We can compute the coefficients by

$$C_k = \frac{1}{N_0} \sum_{N_0} x[n] e^{-jk\Omega_0 n}$$

The DTFT of periodic signals gives a spectrum that is discrete. But what does periodic mean in this context? It basically means that we take the data for one period and perform the analysis only on that. The length of the period results in sampling of the continuous spectrum that we calculated for the DTFT of an aperiodic signals. So in fact we can take any piece of a signal, pretend that it is *a one complete period*, and then go ahead and do the DTFT on this finite length. Such a process gives a discrete spectrum. This is the basis of the next topic, the Discrete Fourier transform (DFT). The DFT is same as the DTFT of periodic signals, except that we use all the data available and call it one period of a presumed periodic signal.

Summary of Chapter 5

In this chapter we examined the Fourier transform for discrete signals, or the DTFT. The DTFT spectrum of aperiodic signals is a continuous function of frequency hence it is not considered a practical tool. It is mostly of theoretical and educational interest and is a bridge topic to DFT. The DTFT of periodic signals however is a sampled version of the continuous DTFT. From the DTFT, we drive the DTFT of finite length signals, called the DFT, in the next chapter.

Terms introduced in this chapter:

- **DTFT** - Discrete-time Fourier transform
- **DFT** - Discrete Fourier transform
- **Dirichlet function** - The periodic version of the sinc function.

1. The DTFT of a discrete-time aperiodic signal is developed by assuming that the period of the discrete pulse is infinitely long, the same idea applied to continuous-time signals to develop the CTFT.
2. Because the period is presumed very long, the frequency resolution approaches zero, hence the DTFT, specified by $X(\Omega)$, becomes a continuous function of frequency.
3. The DTFT of an aperiodic signal is continuous just as the CTFT. However, unlike the CTFT, the DTFT repeats for each range of 2π.
4. The DTFT of an aperiodic signal as a function of the digital frequency Ω is unique only in 2π range.
5. We need to compute the DTFT only in this range, as the DTFT in all other frequency ranges are identical to the principal alias.
6. The DTFT is given by
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$
The frequency Ω is continuous.
7. The iDTFT is given by
$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega_0) e^{j\Omega n} d\Omega$$
8. The DTFT of an aperiodic discrete signal is continuous and repeating.
9. The DTFT of a periodic discrete signal is given by
$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} C_k \delta\left(\Omega - \frac{2\pi k}{N_0}\right).$$

10. The DTFT of a periodic signals is the discrete-time Fourier series coefficients, C_k scaled by 2π and repeating with the sampling frequency of the signal.
11. The period of a discrete signal is given by N_0 or by F_s.
12. The DTFT of a discrete periodic signal is similar to discrete-time Fourier series, DTFSC.
13. The DTFT of periodic signals is a sampled version of the discrete-time Fourier series coefficients. Hence it is discrete and repeats with the sampling frequency.
14. The DTFT of a discrete periodic signal is discrete, with frequency resolution of $2\pi/N_0$ with N_0 equal to the samples per period.
15. DTFT of periodic signals leads us to the Discrete Fourier Transform (DFT) which can be used for finite length signals.

Questions

1. The CTFT of an aperiodic CT signal is different from the DTFT of the same signal sampled with a sampling frequency, Fs in what important manner?
2. The DTFT is periodic with what digital frequency?
3. What is the principal difference between the CTFT and the DTFT?
4. If a signal is sampled with frequency F_s, what is the frequency range of the principal alias?
5. At what sampling frequency will a sinc function appear as a single delta function?
6. A discrete signal has a period of 12 samples. What is its digital frequency?
7. Why does the DTFT repeat?
8. What is the fundamental digital frequency of this periodic sequence. x[n] = [1 0 1 0 0 2]
9. Given the fundamental period, what is the digital frequency of a signal?
10. For a discrete signal with a period of 1001 samples, how many unique basis signals can be used in determining its DTFT.
11. Why do we use digital frequency for discrete signals?
12. What is the DTFT of $x[n] = 1$, $X(\Omega) =$?
13. What is the DTFT of $x[n] = \delta[n-4]$?
14. What is the DTFT of the sequence $x(t) = [\ 1\ 0\ 1\ ]$?
15. Which formulation best specifies the amplitude spectrum of a signal, $x(t)$. $X(f)$; $|X(f)|e^{j\phi}$; $|X(f)|$.
16. State the time scaling property of a signal. A signal is speeded by a factor of 4 in time, what happens to its DTFT?
17. By the sifting property of the delta function, what is the result of this expression; $e^{(-2\omega t)}\delta(t-3)$?
18. What is the fundamental period of the following signal? $x(t)\sin^2(3\omega t) - .5\cos(5\omega t)$

19. The sampling time is related to the fundamental period by what relationship?
20. A signal has bandwidth of 500 Hz. What is the highest and the lowest sampling frequency that we can use?
21. What is the shape of the FT of a Gaussian pulse.
22. A square pulse lasts 1 second. What is it DTFT?
23. If a signal is of bandwidth 100 Hz, and the sampling time is .005 seconds, what is its Nyquist rate.
24. If a signal is being sampled at sampling frequency of 100 Hz, what is its fundamental period.
25. What is the CTFT of a sinusoid of frequency 4 Hz.
26. How does the CTFT of a sine wave differ from the CTFT of a cosine wave.
27. Can you spot the errors in this expression for the iDTFT. $x[n] = \frac{1}{N} \int_{\pi} X(\Omega) e^{j\Omega_0 k} d\Omega$
28. If a function is shifted in frequency domain by 3 radians, what happens to the signal in time domain?
29. If a function is shifted in frequency domain by 2 Hz, by what would its iDTFT be multiplied, per the shift property?
30. What is the main difference between the Diric and the Sinc function?
31. The CTFT of a periodic signal is different from the CTFSC in what way?
32. The DTFT of a periodic signal is different from the DTFSC in what way?

Table 5.2: DTFT of common signals

Signal $x[n]$	DTFT $X(\Omega)$
$1,\ -\infty < n < \infty$	$X(\Omega) = 2\pi \sum_{n=-\infty}^{\infty} \delta(\Omega - 2\pi k)$
$\mathrm{sgn}[n] = \begin{cases} -1 & 0 > n \\ 1 & 0 \leq n \end{cases}$	$\dfrac{1}{1 - e^{-j\Omega}}$
$u[n]$	$\dfrac{1}{1 - e^{-j\Omega}} + 2\pi \sum_{n=-\infty}^{\infty} \delta(\Omega - 2\pi k)$
$\delta[n]$	$1,\ -\infty < \Omega < \infty$
$\delta[n - n_0]$	$e^{-j\Omega n_0}$
$a\delta[n - n_1] + b\delta[n - n_2]$	$ae^{-j\Omega n_1} + be^{-j\Omega n_2}$
$a^n u[n],\ \lvert a \rvert < 1$	$\dfrac{1}{1 - ae^{-j\Omega}}$
$e^{j\Omega_0 n},\ \Omega_0$ real	$X(\Omega) = 2\pi \sum_{n=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k)$
Square pulse of width τ centered at $n = 0$	$\dfrac{\sin(\tau\Omega/2)}{\sin(\Omega/2)}$
Square pulse of width τ centered at $n = n_0$	$\dfrac{\sin(\tau\Omega/2)}{\sin(\Omega/2)} e^{-j\Omega n_0}$
$\cos(\Omega_0 n)$	$\pi \sum_{n=-\infty}^{\infty} \left[\delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k)\right]$
$\cos(\Omega_0 n + \phi)$	$\pi \sum_{n=-\infty}^{\infty} \left[e^{j\phi}\delta(\Omega - \Omega_0 - 2\pi k) + e^{-j\phi}\delta(\Omega + \Omega_0 - 2\pi k)\right]$
$\sin(\Omega_0 n)$	$-j\pi \sum_{n=-\infty}^{\infty} \left[\delta(\Omega - \Omega_0 - 2\pi k) - \delta(\Omega + \Omega_0 - 2\pi k)\right]$
$\sin(\Omega_0 n + \phi)$	$-j\pi \sum_{n=-\infty}^{\infty} \left[e^{j\phi}\delta(\Omega - \Omega_0 - 2\pi k) - e^{-j\phi}\delta(\Omega + \Omega_0 - 2\pi k)\right]$