# The Intuitive Guide to Fourier Analysis & Spectral Estimation

# with MATLAB<sup>®</sup>

This book will deepen your understanding of Fourier analysis making it easier to advance to more complex topics in digital signal processing and data analysis in mathematics, physics, astronomy, bio-sciences, and financial engineering. With numerous examples, detailed explanations, and plots, we make the difficult concepts clear and easy to grasp.

Fourier transform developed slowly, from the Fourier series 200 years ago to Fourier transform as implemented by the FFT today. We tell you this story, in words and equations and help you understand how each step came about.

- We start with the development of Fourier series using harmonic sinusoids to represent periodic signals in continuous and discrete-time domains.
- From here, we examine the complex exponential to represent the Fourier series basis functions.
- Next, we describe the development of the continuous-time and discrete-time Fourier transforms (CTFT, DTFT) for non-periodic signals.
- We show how the DTFT is modified to develop the Discrete Fourier Transform (DFT), the most practical type of the Fourier transform.
- We look at the properties and limitations of the DFT and its algorithmic cousin, the FFT. We examine the use of Windows to reduce leakage effects due to truncation.
- We examine the application of the DFT/FFT to random signals and the role of auto-correlation function in the development of the power spectrum.
- Lastly, we discuss methods of spectral power estimation. We focus on non-parametric power estimation of stationary random signals using the Periodogram and the Autopower.

If you would like to get updates and notices of error corrections to this book, please email me at charldsp@gmail.com.

And if this book has helped you, please do post a review on Amazon for us. Thank you. Charan Langton and Victor Levin The Intuitive Guide to Fourier Analysis & Spectral Estimation

with MATLAB<sup>®</sup>

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#### The Intuitive Guide to Fourier Analysis and Spectral Estimation

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# **Chapter 3**

# Discrete-Time Signals and Fourier Series Representation



Peter Gustav Lejeune Dirichlet 1805 – 1859

Johann Peter Gustav Lejeune Dirichlet was a German mathematician who made deep contributions to number theory, and to the theory of Fourier series and other topics in mathematical analysis; he is credited with being one of the first mathematicians to give the modern formal definition of a function. In 1829, Dirichlet published a famous memoir giving the conditions, showing for which functions the convergence of the Fourier series holds. Before Dirichlet's solution, not only Fourier, but also Poisson and Cauchy had tried unsuccessfully to find a rigorous proof of convergence. The memoir introduced Dirichlet's test for the convergence of series. It also introduced the Dirichlet function as an example that not any function is integrable (the definite integral was still a developing topic at the time) and, in the proof of the theorem for the Fourier series, it introduced the Dirichlet kernel and the Dirichlet integral. – From Wikipedia

# Fourier Series and Discrete-time Signals

In the previous two chapters, we discussed the Fourier series as applied to CT signals. We saw that the Fourier series can be used to create a representation of any periodic signal. This representation is made using the sine and cosine functions or with complex exponentials. Both forms are equivalent. In the previous two chapters, the discussion was limited to continuous time (CT) signals. In this chapter we will discuss Fourier series analysis as applied to discrete time (DT) signals.

#### Discrete signals are different

Although some data are naturally discrete such, as stock prices, number of students in a class, etc., many electronic signals we work with are sampled from analog signals, for example, voice, music, and medical/biological signals. The discrete signals are generated from analog signals by a process called **sampling**. This is also known as **Analog-to-Digital** conversion. The generation of a discrete signal from an analog signal is done by an instantaneous measurement of the analog signal amplitude at uniform intervals.

#### Discrete vs. digital signals

In general terms, a **discrete signal** is *continuous in amplitude* but is *discrete in time*. This means that it can have any value whatsoever for its amplitude but is defined or measured only at *uniform* time intervals. Hence, the term *discrete* applies to the time dimension and not to the *amplitude*. For purposes of the Fourier analysis, we assume that the sampling is done at uniform time intervals among the samples.

A discrete signal is often confused with the term **digital signal**. Although in common language they are thought of as the same thing, a digital signal is a special type of discrete signal. Like any discrete signal, it is defined only at specific time intervals, but its amplitude is constrained to specific values. There are binary digital signals where the amplitude is limited to only two values,  $\{+1, -1\}$  or  $\{0, 1\}$ . A *M*-level signal can take on just one of  $2^M$  preset amplitudes only. Hence, a *digital* signal is a specific type of discrete signal with *constrained amplitudes*. In this chapter, we will be discussing general discrete signals that include *digital signals*. Both of these types of signals are called discrete time (DT) signals. We call the time of a sampling event, the *sampling instant*. How fast or slow a signal is sampled is specified in terms of its *sampling frequency*, which is given in terms of the number of samples per second captured.



Figure 3.1: Discrete sampling collects the actual amplitudes of the signal at the sampling instant, whereas digital sampling rounds the values to the nearest allowed value. In (b), the sampling values are limited to just 2 values, +1 or -1. Hence, each value from (a) has been rounded to either a +1 or -1 to create a binary digital signal.

#### Generating discrete signals

In signal processing, we often need to distinguish between a CT and a DT signal. First way to distinguish the two is to note that we use letter n as the index of discrete time for a DT signal, whereas we use letter t for the index of time for a CT signal. The second way is that a DT signal is written with square brackets around the time index, n, whereas the CT signal is written with round brackets around the time index, t. The two types, CT and DT, are written as follows:

x(t) A continuous time signal x[n] A discrete time signal

We can create a discrete signal by multiplying a continuous signal with a sampling signal, as shown in Fig. 3.2(b). This type of signal is called an impulse train and has a mathematical equation in terms of an infinite number of delta functions located at uniform intervals. This is a very special type of sampling function that is not only easy to visualize, but is also considered the ideal sampler. We give it a generic designation of p(t) for the following discussion. The sampled function,  $x_s$ , is a simply the product of the CT signal and the sampling function, p(t). We write the sampled function as:

$$x_s(t) = x(t)p(t) \tag{3.1}$$



Figure 3.2: A CT signal sampled at uniform intervals  $T_s$  with an ideal sampling function. The discrete signal in (c) x[n] consists only of the discrete samples and nothing else. The continuous signal is shown in dashed line for reference only. The receiver has no idea what it is. All it sees are the samples.

## Sampling and Interpolation

#### Ideal sampling

Let us assume we have an impulse train, p(t) with period  $T_s$  as the sampling function. Multiplying this signal with the CT signal, as shown in Eq. (3.1), we get a continuous signal with nonzero samples at the sample instants, referred to as  $nT_s$  or  $n/F_s$ . Hence, the absolute time is the sample number times the time in between each sample.

The sample time  $T_s$  is an independent parameter. The inverse of the sampling time,  $T_s$ , is called the **sampling frequency**,  $F_s$ , given in samples per second. For a signal sampled with the ideal sampling function, an impulse train, the sampled signal is written per Eq. (3.1) as:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$
$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$
(3.2)

The expression for a discrete signal of a sampled version of the CT signal is written as:

$$x[n] = x_s(t)|_{t=nT_s} = x(nT_s)$$
(3.3)

The term  $x(nT_s)$  with round brackets is continuous, as it is just the value of the CT signal at time  $nT_s$ . The term x[n], however, is discrete because the index n is an integer by definition. The discrete signal x[n] has values only at points  $t = nT_s$  where n is the integer sample number. It is undefined at all non-integers *unlike* the CT signal. The sampling time  $T_s$  relative to the signal frequency determines how coarse or fine the sampling is. The discrete signal can of course be real or complex. The individual value x[n] is called the *n*th sample of the sequence.

If we are given a CT signal of frequency  $f_0$ , and this is being sampled at M samples per second, we would compute the discrete signal from the continuous signal with this Matlab code. Here, time *t* has been replaced with  $n/F_s$ .

```
1 xc = sin(2*pi*f0*t)
2 Fs = 24
3 n = -48: 47
4 xd = sin(2*pi*n/Fs)}
```

### Reconstruction of an analog signal from discrete samples

Why sample signals? The signals are sampled for one big reason, to reduce their bandwidth. The other benefit from sampling is that signal processing on digital signals is "easier." However, once sampled, processed, and transmitted, this signal must often then be converted back to its analog form. The process of reconstructing a signal from discrete samples is called **interpolation**. This is the same idea as plotting a function. We compute a few values at some selected points and then connect those points to plot the continuous representation of the function. The reconstruction by machines, however, is not as straightforward and requires giving them an algorithm that they are able to do. This is where the subject gets complicated.

First, we note that there are two conditions for ideal reconstruction. One is that the signal must have been ideally sampled to start with, i.e., by an impulse train such that the sampled values represent true amplitudes of a signal. Ideal sampling is hard to achieve but for textbook purposes, we assume it can be done. In reality, lack of ideal sampling introduces distortions.

The second is that the signal must not contain any frequencies above one-half of the sampling frequency. This second condition can be met by first filtering the signal by an antialiasing filter, a filter with a cutoff frequency that is one-half the sampling frequency prior to sampling. Alternatively we can assume that the sampling frequency chosen is large enough to encompass all the important frequencies in the signal. Let us assume this is also done.



Figure 3.3: We can reconstruct the analog signal by replacing each sample by one of these shapes on the left: rectangular, triangular, or the complicated-looking sinc function. Each has a distinct frequency response as shown on the right.

For the purposes of reconstruction, we chose an arbitrary pulse shape, h(t). The idea is that we will replace each discrete sample with this pulse shape, and we are going to do this by convolving the pulse shape with the sampled signal. Accordingly, the sampled signal  $x(nT_s)$  (which is same as x[n]) convolved with an arbitrary shape, h(t), is written as:

$$x_r(t) = \left\{ \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \right\} * h(t)$$
(3.4)

The subscript r in Eq. (3.4) indicates that this is a reconstructed signal. At each sample n, we convolve the sample (a single value) by h(t), a little wave of some sort, lasting some time. This convolution in Eq. (3.4) centers the "little wave" at the sample location. All these packets of waves are then arrayed and added in time. (Note that they are continuous in time.) Depending on the h(t) or the little wave selected, we get a reconstructed signal which may or may not be a good representation of the original signal.

Simplifying this equation by completing the convolution of h(t) with an impulse train, we write this somewhat simpler equation for the reconstructed signal as:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)h(t - nT_s)$$
(3.5)

To examine the possibilities for shapes, h(t), following three options are picked; a rectangular pulse, a triangular pulse, and a sinc function. It turns out that these three pretty much cover most of what is used in practice. Each of these "shapes" has a distinctive frequency response as shown in Fig. 3.3. The frequency response is used to determine the effect these shapes will have on the reconstructed signal. Of course, we have not yet fully explained what a *frequency response* is. A frequency response is meant to identify or characterize physical systems. It is done by injecting an impulse into a system and then noting the output. This output is called the **frequency response** of the system. It is often characterized by the magnitude of the response, and the phase,

so it is very similar to the idea of a spectrum. In industry these two terms are used interchangeably.

The sampled signals often require that we reconvert them back to their analog form. We discuss three main ways this is done. Intuitively speaking, the process consists of replacing each sample with a little wave.

#### Method 1: Zero-Order-Hold

Figure 3.3(a) shows a single square pulse. The idea is to replace each sample with a square pulse of amplitude equal to the sample value. This basically means that the sample amplitude is held to the next sampling instant in a flat line. The hold time period is  $T_s$ . This form of reconstruction is called **sample-and-hold** or **zero-order-hold (ZOH)** method of signal reconstruction. Zero in ZOH is the slope of this interpolation function, a straight line of zero slope connecting one sample to the next. It is a simplistic method but if done with small enough resolution, that is a very narrow rectangle in time, ZOH can do a decent job of reconstructing the signal. The shape function h(t) in this case is a rectangle.

$$h(t) = \operatorname{rect}(t - nT_s) \tag{3.6}$$

The reconstructed signal is now given using the general expression of Eq. (3.4), where we substitute the [rect] shape into Eq. (3.5) to get:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{rect}(t - nT_s)$$
(3.7)

We show the index as going from  $-\infty < n < +\infty$  as the general form. In Fig. 3.4(a), we see a signal reconstructed using a ZOH circuit, in which the rectangular pulse is scaled and repeated at each sample.

#### Method 2: First-Order-Hold (linear interpolation)

Zero-order hold gives us a stair-step like signal. Now, instead of a square pulse, we replace each sample with a triangle of width  $2T_s$  as given by the expression

$$h(t) = \begin{cases} 1 - t/T_s & 0 < t < T_s \\ 1 + t/T_s & T_s < t < 2T_s \\ 0 & \text{else} \end{cases}$$
(3.8)



Figure 3.4: A zero-order-hold: hold each sample value to the next sample time.



Figure 3.5: A First-order-hold (FOH): replace each sample with a triangle of twice the time-width.

This function is shaped like a triangle and the reconstructed signal equation from Eq. (3.5) now becomes

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{tri}\left(\frac{t - nT_s}{T_s}\right)$$
(3.9)

Figure 3.5(b) shows that instead of nonoverlapping rectangles as in ZOH, we use overlapping triangles. This is because the width of the triangle is set to twice the sample time. This double-width does two things: it keeps the amplitude the same as the case of the rectangle and it fills the in-between points in a linear fashion. This method is also called a **linear interpolation** as we are just connecting the points. This is also called the **First-order-hold** (FOH) because we are connecting the adjacent samples with a line of linear slope. Why you may ask use triangles when we can just connect the samples? The answer to this query is that machines cannot "see" the samples nor "connect" the samples. Addition is about all they can do well. Hence, this method replaces the linear interpolation as you and *I* might do, visually with a simple addition of displaced triangles. It also gives us a hint as to how we can use any shape we want and in fact of any length, not just two times the sample time! Sinc pulse is such a shape.

#### Method 3: Sinc interpolation

We used triangles in FOH and that seems to produce a better-looking reconstructed signal than ZOH. We can, in fact use just about any shape we want to represent a sample, from the rectangle to one composed of a complex shape, such as a sinc function. A sinc function seems like an unlikely choice as it is noncausal (as it extends into the future) but it is infact an extension of the idea of the first two methods. Both ZOH and FOH are forms of polynomial curve fit. The FOH is a linear polynomial, and we continue in this fashion with second-order on up to infinite orders to represent just about any type of wiggly shape we can think of. A sinc function, an infinite order polynomial, is the basis of perfect reconstruction. The reconstructed signal becomes a sum of scaled, shifted sinc functions same as was done with triangular shapes. Even though the sinc function is an infinitely long function, it is zero-valued at regular intervals. This interval is equal to the sampling period. As each sinc pulse lobe crosses zero at only the sampling instants, the summed signal where each sinc is centered at a different time, adds no interference (quantity of its own amplitude) to other sinc pulses centered (ISI).

The equation obtained for the reconstructed signal in this case is similar to the first two cases, with the reconstructed signal summed with each sinc located at  $nT_s$ .

$$h(t) = \operatorname{sinc}(t - nT_s)$$

$$x_r(t) = \sum_{n = -\infty}^{\infty} x(nT_s)\operatorname{sinc}(t - nT_s)$$
(3.10)

Figure 3.6 shows the sinc reconstruction process for a signal with each sample being replaced by a sinc function and the resulting reconstructed signal compared to the original signal in Fig. 3.6(b).

Clearly the sinc construction in Fig. 3.6(c) does a very good job. How to tell which of these three methods is better? The ZOH is kind of rough. But to properly assess these methods, we require a full understanding of the Fourier transform, a topic that will be covered in Chapter 4. Therefore, we will drop this subject now with recognition that a signal can be reconstructed using the linear superposition principal using many different shapes, with sinc function being one example, albeit a really good one, the one we call the "perfect reconstruction."

#### Sinc function detour

We will be coming across the sinc function a lot. It is not only the most versatile, but also most used piece of mathematical concept in signal processing. Hence, we examine the sinc function in a bit more detail here. In Fig. 3.7, the function is plotted



Figure 3.6: Sinc reconstruction: replace each sample with a sinc pulse

in time-domain. This form is called the *normalized sinc function*. The sinc function is a continuous function of time, *t*, and is not periodic. It is that nice-looking single-peak signal that oscillates and eventually damps out.

$$h(t) = \begin{cases} 1 & t = 0\\ \frac{\sin(\pi t)}{\pi t} & t \neq 0 \end{cases}$$
(3.11)

At t = 0, its value is 1.0. As seen in Eq. (3.11), the function is zero for all integer values of t, because sine of an integer multiple of  $\pi$  is zero. In Matlab, this function is given as sinc(t). No  $\pi$  is needed as it is already programmed in. The Matlab plot would yield first zero crossing at  $\pm 1$  and as such the width of the main lobe is 2 units. By inserting a variable  $T_s$  into equation Eq. (3.12), any main lobe width can be created. The generic sinc function of lobe width  $T_s$  (main lobe width  $= 2T_s$ ) is given by

$$h(t) = \frac{\sin(\pi t/T_s)}{\pi t/T_s}$$
(3.12)

```
A sinc function in two forms
1
  응
           -6: .01: 6;
       t
2
         =
          = 2;
       Τs
3
        = sinc(t/Ts);
4
       h
       habs = abs(h1)
5
       plot(t, h, t, habs)
6
```



Figure 3.7: The sinc function in time-domain. The signal is nonperiodic. Its peak value is 1 for the normalized form. The main lobe of the function spans two times the parameter T<sub>s</sub>. In (b) the absolute values are shown so the lobes on the negative side are flipped up.

In Matlab, we would create a sinc function as in the code above.

Figure 3.7 shows this function for two different values of,  $T_s = 2$  and  $T_s = 1$  (same as the normalized case). The sinc function is often plotted in the second style, Fig. 3.7(b) with amplitudes shown as absolute values. This style makes it easy to see the lobes and the zero crossings. Note that zero-crossings occur every  $T_s$  seconds. Furthermore the function has a main lobe that is 2 times  $T_s$  seconds wide. All the other lobes are  $T_s$  seconds wide.

The sinc function has some interesting and useful properties. The first one is that the area under it is equal to 1.0.

$$\int_{-\infty}^{\infty} \operatorname{sinc}(2\pi t/T_s) \, dt = \operatorname{rect}(0) = 1. \tag{3.13}$$

The second interesting and useful property, from Eq. (3.12) is that as  $T_s$  decreases, the sinc function approaches an impulse. This is seen in Fig. 3.7. A smaller value of  $T_s$  means a narrower lobes. Narrow main lobe makes the central part impulse-like and hence it is noted that as  $T_s \rightarrow 0$  goes to zero, the function approaches an impulse. Another interesting property is that the sinc function is equivalent to the summation of all complex exponentials. This is a magical property in that it tells us how Fourier transform works by scaling these exponentials. We have shown this effect in Chapter 1 by adding many harmonics together and noting that the result approaches an impulse train.

$$\operatorname{sinc}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega t} \, d\omega \tag{3.14}$$

This property is best seen in Fig. 3.8, which shows the result of an addition of a large number of harmonic complex exponentials together. The signal looks very much like an impulse train.



Figure 3.8: The addition of 90 consecutive harmonics (f = 1, 2, ..., 90), result in very nearly an impulse train.

The sinc function is also the frequency response to a square pulse. It can be said that it is a representation of a square pulse in the frequency domain. If a square pulse (also called a rectangle, probably a better name anyway) is taken in time domain, then its Fourier series representation will be a sinc, alternately, the sinc function has a frequency representation of a rectangle, which says that it is absolutely bounded in bandwidth. We learn from this that a square pulse in time domain has very large (or in fact infinite) bandwidth and is not a desirable pulse type to transmit.

#### Sampling rate

How do we determine an appropriate sampling rate for an analog signal? Figure 3.9 shows an analog signal sampled at two different rates; the signal is sampled slowly and sampled rapidly. At this point, our idea of slow and rapid is arbitrary.



Figure 3.9: The sampling rate is an important parameter, (a) analog signal sampled probably too slowly, (b) probably too fast.

It is obvious by looking at the samples in Fig. 3.9(a) that the rate is not quick enough to capture all the ups and downs of the signal. Some high and low points have been missed. However the rate in Fig. 3.9(b) looks like it might be too fast as it is capturing more samples than we may need. Can we get by with a smaller rate? Is there an

optimum sampling rate that captures just enough information such that the sampled analog signal can still be reconstructed faithfully from the discrete samples?

# Shannon's Theorem

There is an *optimum sampling rate*. This optimum sampling rate was established by Harry Nyquist and Claude Shannon and others before them. However the theorem has come to be attributed to Shannon and is thus called the **Shannon's theorem**. Although Shannon is often given credit for this theorem, it has a long history. Even before Shannon, Harry Nyquist (a Swedish scientist who immigrated to USA in 1907 and did all his famous work in the USA) had already established the **Nyquist rate**. Shannon took it further and applied the idea to reconstruction of discrete signals. And even before Nyquist, the sampling theorem was used and specified in its present form by a Russian scientist **V. A. Kotelnikov** in 1933. In fact even he may have not been the first. So simple and yet so profound, the theorem is a very important concept for all types of signal processing.

The theorem says:

For any analog signal containing among its frequency content a maximum frequency of  $f_{\text{max}}$ , then the analog signal can be represented faithfully by N equally spaced samples, provided the sampling rate is at least two times  $f_{\text{max}}$  samples per second.

We define the **Sampling frequency**  $F_s$ , as the number of samples collected per second. For a faithful representation of an analog signal, the sampling rate  $F_s$  must be equal or greater than two times the maximum frequency contained in the analog signal. We write this rule as

$$F_s \ge 2f_{\max} \tag{3.15}$$

The **Nyquist rate** is defined as the case of sampling frequency  $F_s$  exactly equal to two times  $f_{\text{max}}$ . This is also called the **Nyquist threshold** or **Nyquist frequency**.  $T_s$  is defined as the time period between the samples, and is the inverse of the sampling frequency,  $F_s$ .

A real life-signal will have many frequencies. In setting up the Fourier series representation, we defined the lowest of all its frequencies,  $f_0$ , as its *fundamental frequency*. The fundamental period of the signal,  $T_0$ , is the inverse of the fundamental frequency as defined in chapter 1.

The *maximum frequency*,  $f_{max}$ , contained within the signal is used to determine an appropriate sampling frequency for the signal,  $F_s$ . An important thing to note is that the fundamental frequency,  $f_0$ , is not related to the maximum frequency of the signal. Hence, there is no relationship *whatsoever* between the fundamental frequency,  $f_0$ , of



Figure 3.10: There is no relationship between the sampling period and the fundamental period of the signal. They are independent quantities.

the analog signal, the maximum frequency,  $f_{max}$ , and the sampling frequency,  $F_s$ , picked to create a discrete signal from the analog signal. The same is true for the fundamental period,  $T_0$ , of the analog signal and the sampling period  $T_s$ . They are not related either. This point can be confusing.  $T_0$  is a property of the signal, whereas  $T_s$  is something chosen externally for sampling purposes. The maximum frequency similarly indicates the bandwidth of the signal, that is  $f_{max} - f_0$ .

The Shannon theorem applies, strictly speaking, only to baseband signals or the low-pass signals. There is a complex-envelope version where even though the center frequency of a signal is high due to having been modulated and up-converted to a higher carrier frequency, the signal can still be sampled at twice its bandwidth and be perfectly reconstructed. This is called the band-pass sampling theorem. This will not be taken up in this book.

#### Aliasing of discrete signals

Figure 3.11(a) shows discrete samples of a signal whereas Fig. 3.11(b) shows that these points fit several waves shown. So which wave did they really come from?



Figure 3.11: Three signals of frequency 1, 3 and 5 Hz all pass through the same discrete samples shown in (a). How can we tell which frequency was transmitted?

The samples in Fig. 3.11(a) could have, in fact, come from an infinite number of other waves which are not shown. This is a troubling property of discrete signals. This effect, that many different frequencies can be mapped to the same samples, is called **alias-ing**. This effect, caused by improper sampling of the analog signal, leads to erroneous conclusions about the signal. Later, we will discuss how the spectrum of a discrete signal repeats, and it repeats precisely for this reason that we do not know the real frequency of the signal. Its a way of the math telling us that it does not know the real answer among many seemingly correct solutions.

#### **Bad sampling**

If a sinusoidal signal of frequency  $f_0$  (because a sine wave only has one frequency, both its highest and its lowest frequencies are the same) is sampled at less than two times the maximum frequency,  $F_s < 2f_0$ , then the signal that is reconstructed, although passing through all the samples, is erroneously mapped to a wrong frequency. This wrong frequency, an *alias*, is not the one that we started with. Given a sampling frequency, we can identify all possible alias frequencies by this expression.

$$y(t) = \sin(2\pi (f_0 - mF_s)t)$$
(3.16)

Here, m is a positive integer satisfying this equation

$$\left|f_0 - mF_s\right| \le \frac{F_s}{2} \tag{3.17}$$

These two equations are very important but they are not intuitive. So let us take a look at an example.

**Example 3.1.** Take the signal with  $f_0 = 5$  Hz and  $F_s = 8$  Hz or 8 samples per second or samps. We use Eq. (3.16) to find the possible alias frequencies. Here are first three for (m = 1, 2, 3, ...) aliases.

$$m = 1: y(t) = \sin(2\pi(5 - 1 \times 8)t) = \sin(2\pi\underline{3}t)$$
  

$$m = 2: y(t) = \sin(2\pi(5 - 2 \times 8)t) = \sin(2\pi\underline{11}t)$$
  

$$m = 3: y(t) = \sin(2\pi(5 - 3 \times 8)t) = \sin(2\pi\underline{19}t)$$

The first three alias frequencies computed are 3, 11, and 19 Hz, all varying by 8 Hz, the sampling frequency. The samples fit all of these frequencies. The significance of m, the order of the aliases is as follows. When the signal is reconstructed, we need to filter it by an antialiasing filter to remove all higher frequency aliases. Setting m = 1 implies the filter is set at frequency of  $F_s/2$  or in this case 4 Hz, as per the limitation set on index m by Eq. (3.17). Therefore, we only see alias frequencies that are below this number.

Higher order aliases although present are not seen. In computing the possible set of alias frequencies, the value of m is limited by Eq. (3.17).

Figure 3.12 shows the Eq. (3.16) in action. Each m in this expression represents a shift. For m = 1, the cutoff point is 4 Hz, which only lets one see the 3 Hz alias frequency but not 11 Hz or higher.



Figure 3.12: The spectrum of the signal repeats with sampling frequency of 8-Hz. Only the 3-Hz signal is below the 4-Hz cutoff. Note that each pair has the same distance between the pulses, but each pair is displaced from the other pair by the sampling frequency.

The fundamental pair of components (the real signal before reconstruction) are at +5 and -5 Hz. Now from Eq. (3.16), this spectrum (the bold pair of impulses at ±5 Hz) repeats with a sampling frequency of 8 Hz. Hence, the pair centered at 0 Hz is also centered at 8 Hz (dashed lines). The lower component falls at 8-5=3 Hz and the upper one at 8+5=13 Hz. The second shift centers the components at 16 Hz, with lower component at 16-5=11 Hz and the higher at 16+5=21 Hz. The same thing happens on the negative side. All of these are called alias pairs. They are all there unless the signal is filtered to remove these. However, the limit on what we can see is placed by the sampling frequency and Eq. (3.17). A system of sampling frequency of 8 Hz will allow us to detect only the one incorrect alias of 3 Hz as shown in highlighted part of Fig. 3.19.

#### Good sampling

The sampling theorem states that you must sample a signal at twice or higher times its maximum frequency to properly reconstruct it from its samples. The consequence of not doing this is we get aliases (from Eq. (3.16)) at wrong frequencies. But what if we do sample at twice or greater rate. Does that have an effect and what is it?

Example 3.2.

$$x(t) = 0.2\sin(2\pi t) + \sin(4\pi t) + 0.7\cos(6\pi t) + 0.4\cos(8\pi t)$$

Let us take this signal as shown in Fig. 3.13(a). The signal has four frequencies, which are 1, 2, 3 and 4 Hz. The highest frequency is 4 Hz. We sample this signal at 20 Hz and also at 10 Hz. Both of these frequency choices are above the Nyquist rate, so that is good.

The spectrum as computed by the FSC of the four frequencies in this signal is shown in Fig. 3.13(b). (We have not yet discussed how to compute this discrete spectrum and will do so soon, but the idea is the same as for the CT case.)

A very important fact for discrete signals is that the FSC repeat with integer multiple of the sampling frequency  $F_s$ . The entire spectrum is copied and shifted to a new center frequency to create an *alias* spectrum. This theoretically continues forever on both sides of the principal alias, shown in a dashed box in the center in Fig. 3.14. The spectrum centered at the zero frequency is called the *Principal alias*.

The DT version of the CT signal is given by setting CT time t to  $n/F_s$  in the following expression. Here n is the sample number and  $F_s$ , the sampling frequency.

$$x(t) = 0.2\sin(2\pi \frac{n}{F_{c}}) + \sin(4\pi \frac{n}{F_{c}}) + 0.7\cos(6\pi \frac{n}{F_{c}}) + 0.4\cos(8\pi \frac{n}{F_{c}})$$

Figure 3.13 shows the signal sampled at 20 Hz, and we see that there is plenty of



Figure 3.13: (a) A composite signal of several sinusoid is sampled at twice the highest frequency. (b), the discrete coefficients repeating with the sampling frequency,  $F_s = 20$  Hz.



Figure 3.14: The signal sampled at  $F_s = 10$  in (a), results in much closer replications in (b).

distance between the copies. This is because the bandwidth of the signal is only 8 Hz, hence, there is 12 Hz between the copies. Figure 3.14(b) shows the spectrum for the signal when sampled at 10 Hz. The spectrum is 8 Hz wide but now the spectrum are close together with only 2 Hz between the copies.

Decreasing the sampling rate decreases the spacing between the alias spectrum. The copies would start to overlap if they are not spaced at least two times the highest frequency of the signal. In such a case, separation of one spectrum from another becomes impossible. When nonlinearities are present, the sampling rate must be higher than Nyquist threshold to allow the spectrum to spread but not overlap. The same is true for the effect of the roll-off from the antialiasing filter. As practical filters do not have sharp cutoffs, some guard band has to be allowed. This guard band needs to be taken into account when choosing a sampling frequency.

If the spectrum do overlap, the effect cannot be gotten rid of by filtering. As we do not have knowledge of the signal spectrum, we are not likely to be aware of any aliasing if it happens. We always hope that we have correctly guessed the highest frequency in the signal and hence have picked a reasonably large sampling frequency to avoid this problem.

However, often we do have a pretty good idea about the target signal frequencies. We allow for uncertainties by sampling at a rate that is higher than twice the maximum frequency, and usually much higher than twice this rate. For example, take audio signals that range in frequency from 20 to 20,000 Hz. When recording these signals, they are typically sampled at 44.1 kHz (CD), 48 kHz (professional audio), 88.2 kHz, or 96 kHz rates depending on quality desired. Signals subject to nonlinear effects spread in bandwidth after transmission and require sampling rates of 4 to 16 times the highest frequency to cover the spreading of the signal.

# **Discrete Signal Parameters**

There are important differences between discrete and analog signals. An analog signal is defined by parameters of *frequency* and *time*. To retain this analogy of time and frequency for discrete signals, we use n, the sample number as the unit of discrete time. The frequency however, gives us a problem. If in DT, time has units of *sample*, then the frequency of a discrete signal must have units of *radians per sample*.

The frequency of a discrete signal is indeed a different type of frequency than the traditional frequency of continuous signals. It is given a special name of its own, **digital frequency** and we use the symbol  $\Omega$  to designate it. We can show the similarity of this

frequency to an analog frequency by noting first how these two forms of signals are written.

Analog signal : 
$$x(t) = \sin(2\pi f_0 t)$$
  
Discrete signal :  $x[n] = \sin(2\pi f_0 nT_s)$  (3.18)

The first expression is a continuous signal and the second a discrete signal. For the discrete signal, replace CT, t, with  $nT_s$ . Alternately, write the discrete signal, as in Eq. (3.19) by noting that the sampling time is inverse of the sampling frequency. (We *always* have the issue of sampling frequency even if the signal is naturally discrete and was never sampled from a continuous signal. In such a case, the sampling frequency is just the inverse of time between the samples.)

$$x[n] = \sin\left(\frac{2\pi f_0}{F_s}n\right). \tag{3.19}$$

#### Digital frequency, only for discrete signals

We define the *Digital frequency*,  $\Omega$  by this expression.

$$\Omega = \frac{2\pi f_0}{F_s} \tag{3.20}$$

Substitute this definition of digital frequency into Eq. (3.19) to obtain an expression for a sampled sinusoid in discrete time.

$$x[n] = \sin(\Omega n) \tag{3.21}$$

Here are two analogous expressions for a sinusoid, a discrete and a continuous form.

Analog signal : 
$$x(t) = \sin(\omega t)$$
  
Discrete signal :  $x[n] = \sin(\Omega n)$  (3.22)

The digital frequency  $\Omega$  is equivalent in concept to an analog frequency, but these two "frequencies" have different units. The analog frequency has units of **radians per sec-ond**, whereas the digital frequency has units of **radians per sample**.

The fundamental period of a discrete signal is defined as a certain number of samples,  $N_0$ . This is equivalent in concept to the fundamental period of an analog signal,  $T_0$ , given in real time. To be considered periodic, a discrete signal must repeat after  $N_0$  samples. In the continuous domain, a period represents  $2\pi$  radians. To retain equivalence in both domains,  $N_0$  samples hence must also cover  $2\pi$  radians, from which we have this relationship.

$$\Omega_0 N_0 = 2\pi \tag{3.23}$$

The units of the fundamental digital frequency  $\Omega_0$  are radians/sample and units of  $N_0$  are just samples. The digital frequency is a measure of the number of radians the signal moves per sample. Furthermore, when it is multiplied by the fundamental period  $N_0$ , an integer multiple of  $2\pi$  is obtained. Hence, a periodic discrete signal repeats with a frequency of  $2\pi$ , which is the same condition as for an analog signal. The only difference being that analog frequency is defined in terms of time it takes to cover  $2\pi$  radians and digital frequency in terms of samples needed to cover the same.



Figure 3.15: A discrete signal in time domain can be referred by its sample numbers, n (1 to N) or by the digital frequency phase advance. Each sample advances the phase by  $2\pi/N$  radians. In this example, N is 8. In (a), the x-axis is in terms of real time. In (b) the x-axis is in terms of sample identification number or n. In (c) we note the radians that pass between each sample such that total excursion over one period is  $2\pi$ .

There are three ways to specify a sampled signal. In Figure 3.15(a), two periods of a signal are shown. This is a continuous signal; hence, the *x*-axis is in continuous time, *t*. Now, we sample this signal. Each cycle is sampled with eight samples, with a total of 17 samples are shown, numbered from -8 to +8 in Fig. 3.15(b). This is the discrete representation of signal x(t) in terms of samples that are identified by the sample number, *n*. This is one way of showing a discrete signal. Each sample has a sample number to identify it.

The sample number can be replaced with a instantaneous phase value for an alternate way of showing the discrete signal. Figure 3.15(c) shows that there are 8 samples over each  $2\pi$  radians or equivalently a discrete frequency of  $2\pi/8$  radians per sample. This is the digital frequency,  $\Omega_0$  which is pushing the signal forward by these many radians. Each sample moves the signal further in phase by  $\pi/4$  radians from the previous sample, with two cycles or 16 samples covering  $4\pi$  radians. Hence, we can label the samples in radians. Both forms, using *n* or the phase are equivalent but the last form (using the phase) is more common for discrete signals, particularly in text books, however, it tends to be non-intuitive and confusing.

#### Periodicity of discrete signals

Fourier series representation requires a signal to be *periodic*. Therefore, can we assume that a discrete signal, if it is sampled from a periodic signal, is also periodic? The answer is strangely enough, no. Here, we look at the conditions of periodicity for a continuous and a discrete signal.

Continuous signal : 
$$x(t) = x(t+T)$$
  
Discrete signal :  $x[n] = x[n+N]$  (3.24)

This expression says that if the values of a signal repeat after a certain number of samples, N, for the discrete case and a certain period of time, T, for the continuous case, then the signal is periodic. The smallest value of N that satisfies this condition is called the *fundamental period of the discrete signal*. As we use sinusoids as basis functions for the Fourier analysis, let us apply this general condition to a sinusoid. To be periodic, a discrete sinusoid that is defined in terms of the digital frequency and time sample, n, must repeat after N samples, hence, it must meet this condition.

$$\cos(\Omega_0 n) = \cos\left(\Omega_0 (n+N)\right) \tag{3.25}$$

We expand the right-hand-side of Eq. 3.25 using this trigonometric identity:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
(3.26)

To examine under which condition this expression is true. we set:

$$\cos\left(\Omega_0(n+N)\right) = \cos(\Omega_0 n)\cos(\Omega_0 N) - \sin(\Omega_0 n)\sin(\Omega_0 N). \tag{3.27}$$

For Eq. (3.25) to be true, we need the underlined terms on the RHS to be equal to 1 and 0 respectively.

$$\cos(\Omega_0 n) = \cos(\Omega_0 n) \underbrace{\cos(\Omega_0 N)}_{=1} - \sin(\Omega_0 n) \underbrace{\sin(\Omega_0 N)}_{=0}$$
(3.28)

For these two conditions to be true, we must have

$$\Omega_0 N = 2\pi k \text{ or}$$

$$\frac{\Omega_0}{2\pi} = \frac{k}{N}$$
(3.29)

It is concluded that a discrete sinusoid is periodic if and only if its digital frequency is a rational multiple of  $2\pi$  based on the smallest period *N*. This implies that discrete signals are neither periodic for all values of  $\Omega_0$ , nor for all values of *N*. For example, if  $\Omega_0 = 1$ , then no integer value of *N* or *k* can be found to make the signal periodic per Eq. (3.29).

We write the expression for the fundamental period of a discrete and periodic signal as:

$$N = \frac{2\pi k}{\Omega_0} \tag{3.30}$$

The smallest integer *k*, resulting in an integer *N*, gives the fundamental period of the periodic sinusoid, if it exists. Hence, for k = 1, we get  $N = N_0$ .

**Example 3.3.** What is the digital frequency of this signal? What is its fundamental period?

$$x[n] = \cos\left(\frac{2\pi}{5}n + \frac{\pi}{3}\right)$$

The digital frequency of this signal is  $2\pi/5$  because that is the coefficient of time index



Figure 3.16: Signal of Ex. 3.3.

*n*. The fundamental period  $N_0$  is equal to 5 samples that we find using Eq. (3.30) setting k = 1.

$$N_0 = \frac{2\pi}{\Omega_0} = \frac{2\pi}{2\pi/5} = 5$$

Example 3.4. What is the period of this discrete signal? Is it periodic?

$$x[n] = \sin\left(\frac{3\pi}{4}n + \frac{\pi}{4}\right)$$



Figure 3.17: Signal of Ex. 3.4. The period of this signal is 8 samples.

The digital frequency of this signal is  $3\pi/4$ . The fundamental period is equal to

$$N_0 = \frac{2\pi k}{\Omega_0} = \frac{2\pi (k=3)}{3\pi/4} = 8$$
 samples

The period is 8 samples but it takes  $6\pi$  radians to get the same sample values again. As we see, the signal covers three cycles in 8 samples. As long as we get an integer number of samples in any integer multiple of  $2\pi$ , the signal is considered periodic.

Example 3.5. Is this discrete signal periodic?

$$x[n] = \sin\left(\frac{1}{2}n + \pi\right)$$

The digital frequency of this signal is 1/2. Its period from Eq. (3.30) is equal to

$$N = \frac{2\pi k}{\Omega_0} = 4\pi k$$

As k must be an integer, this number will always be irrational; hence, it will never result in repeating samples. The continuous signal is, of course, periodic but as we can see in Fig. 3.18, there is no periodicity in the discrete samples. They are all over the place, with no regularity.



Figure 3.18: Signal of Ex. 3.5 that never achieves an integer number of samples in any integer multiple of  $2\pi$ .

# Discrete Complex Exponentials as Basis of DT Fourier Series

The **CT Fourier series** (CTFS) is written in terms of trigonometric functions or complex exponentials. Because these functions are harmonic and hence orthogonal to each other, both trigonometric and complex exponentials form a basis set for complex Fourier analysis. The coefficients can be thought of as scaling of the basis functions. We are now going to look at the Fourier series representation for DT signals using DT complex exponentials as the basis functions.

A discrete complex exponential is written by replacing *t* in CT domain with *n*, and  $\omega$  with the digital frequency  $\Omega$ . Now, we have these two forms of the CE just as we wrote the two forms of the sinusoid in Eq. (3.22).

Continuous form of a CE :  $e^{j\omega_0 t}$ Discrete form of a CE :  $e^{j\Omega_0 n}$  The discrete form of the fundamental is expanded as follows. We show a *single* CE of digital frequency  $\frac{2\pi}{N}$  for variable *n*, the DT index of time. The harmonic factor *k* has not yet been included in this equation.

$$e^{j\Omega_0 n} = e^{j\left(\frac{2\pi}{N}\right)n} \tag{3.31}$$

#### Harmonics of a discrete fundamental CE

For an analog signal, we define its harmonics by multiplying its frequency directly by a multiplier k. Can we do the same for discrete signals? Do we just multiply the fundamental frequency by the index k, such as  $2k\pi/N$  for all k? Well, no. If a signal has fundamental digital frequency of  $\pi/5$ , then is frequency  $2\pi/5$  the next harmonic? Yes, and no, because this method leads us into problems.

The range of the digital frequency is  $2\pi$ . To obtain its next harmonic, we increment its frequency by *adding an integer multiple of*  $2\pi$  to it. Hence, the frequency of a *k*th harmonic of a discrete signal is  $(\Omega_0 + 2\pi k)$  or  $(\pi/5 + 2\pi) = 11\pi/5$  for k = 1. This is a very important point. The analog and discrete harmonics have equivalent definitions for purposes of the Fourier analysis. We will see, however, that they do not display the same behavior. We cannot use these traditionally defined discrete harmonics for the Fourier analysis.

> Discrete fundamental :  $e^{j(2\pi/N)n}$ Discrete harmonic :  $e^{j(2\pi/N+2\pi k)n}$

#### Repeating harmonics of a discrete signal

Where each and every harmonic of an analog signal looks different, i.e., has higher frequency, shows more ups and downs, etc., the discrete harmonics defined by  $e^{j(2\pi/N+2\pi k)n}$  for each k are not distinct from each other. They are said to be non-distinct for each harmonic index, k. This is easy to see from this proof.

$$\phi_{k}[n] = e^{j(2\pi/N + 2\pi k)n}$$
  
=  $e^{jk(2\pi/N)n} \underbrace{e^{j(2\pi kn)}}_{=1}$  (3.32)

Note that the second part can be written as in Eq. (3.33) and is equal to 1.0 because the cosine wave at any integer  $2k\pi$  radians is always 1 and the sine for the same is always 0.

$$e^{j(2k\pi n)} = \cos(2kn\pi)_{=1} + j\sin(2kn\pi)_{=0} = 1$$
(3.33)

Each increment of the harmonic by  $2\pi k$  causes the harmonic factor to cancel and result is we get right back to the fundamental! Hence this method of getting at distinct harmonics is for naught!

**Example 3.6.** Show the first two harmonics of an exponential of frequency  $\pi/6$  if it is being sampled with a sampling period of 0.25 s.

The discrete frequency of this signal is  $\pi/6$ . For an exponential given by  $e^{-j\omega_0 t}$ , we replace  $\omega_0$  with  $\pi/6$  and t with n/4. ( $T_s = 0.25$  hence,  $F_s = 4$ ) We write this discrete signal as:

$$x[n] = e^{-j\frac{2\pi}{24}n}$$

Let us plot this signal along with its next two harmonics, which are:

Fundamental : 
$$e^{-j\left(\frac{\pi}{24}\right)n}$$
  
Harmonic 1 :  $e^{-j\left(\frac{\pi}{24}+2(k=1)\pi\right)n} = e^{-j\left(\frac{\pi}{24}+2\pi\right)n}$  (3.34)  
Harmonic 2 :  $e^{-j\left(\frac{\pi}{24}+2(k=2)\pi\right)n} = e^{-j\left(\frac{\pi}{24}+4\pi\right)n}$ 

We plot all three of these in Fig. 3.19. Why is there only one plot in this figure? Simply because the three signals from Eq. (3.34) are identical and indistinguishable.



Figure 3.19: Signal of Example 3.6 the imaginary part is a sine wave, (b) the real part, which is of course a cosine. The picture is same for all integer values of k.

This example demonstrates that for a discrete signal the concept of harmonic frequencies does not lead to meaningful harmonics. All harmonics are the same. But then how can we do Fourier series analysis on a discrete signal if all basis signals are *identical*? So far we only looked at discrete signals that differ by a phase of  $2\pi$ . Although the harmonics obtained this way are harmonic in a mathematical sense, they are pretty much useless in the practical sense, being non-distinct. So where are the distinct harmonics that we can use for Fourier analysis?

The secret hiding place of the discrete harmonics is *inside* the  $2\pi$  range, *N* unique harmonics, perfectly suitable for Fourier analysis. These *N* sub-frequencies are indeed distinct. But there are only *N* of them, with *N* being the fundamental period in number of samples.

Given a discrete signal of period N, the signal has N unique harmonics. Each such harmonic frequency is given by

$$\Omega_k[n] = k \frac{2\pi}{N} n \quad \text{for } k = 0, 1, \dots, N - 1.$$
(3.35)

Increasing *k* beyond N - 1 will give the same harmonic as for k = 0 again. This is of course equivalent to a full  $2\pi$  radian phase traversal. Hence discrete harmonics are obtained not by an increment of  $2\pi$  but by the digital frequency itself, which is a small portion of  $2\pi$ .

**Example 3.7.** Let us check this idea as digital frequency of a signal is varied just within the 0 to  $2\pi$  range instead of as integer increment of  $2\pi$ . Take this signal:

$$x[n] = e^{j\frac{2\pi}{6}n}$$

Its digital frequency is  $2\pi/6$  and its period is equal to 6 samples. We now know that the signals of digital frequencies  $2\pi/6$  and  $14\pi/6$  (which is  $2\pi/6+2\pi$ ) are exactly the same. So, the digital frequency is increased, not by  $2\pi$  but instead in six steps, each time increasing by  $2\pi/6$  so that after six steps, the total increase will be  $2\pi$  as we go from  $2\pi/6$  to  $14\pi/6$ . We do not jump from  $2\pi/6$  to  $14\pi/6$  but instead move in between. We can start with zero frequency or from  $2\pi/6$  or  $2\pi$  as it makes no difference where you start. Starting with 0th harmonic, if we move in six steps, we get these six unique signals.

$$\phi_0 = 2\pi (k = 0)/6 = 0$$
  

$$\phi_1 = 2\pi (k = 1)/6 = 2\pi/6$$
  

$$\phi_2 = 2\pi (k = 2)/6 = 4\pi/6$$
  

$$\vdots$$
  

$$\phi_5 = 2\pi (k = 5)/6 = 10\pi/6$$

The variable k, steps from 0 to K-1, where K is used as the total number of harmonics. Index n remains the index of the sample or time. Note that since the signal is periodic with  $N_0$  samples, K is equal to  $N_0$ .

The process can be visualized as shown in Fig. 3.20 for N = 6. This is our not-so-secret set of N harmonics (within any  $2\pi$  range) that are unique and used as the basis set for discrete Fourier analysis.



*Figure 3.20: Discrete harmonic frequencies in the range of*  $2\pi m$  *to*  $2\pi(m+1)$ *.* 

Figure 3.21 plots these discrete complex exponentials so we can examine them. There are two columns in this figure, with left containing the real and right the imaginary part, together representing the complex-exponential harmonic. The analog harmonics are shown in dashed lines for elucidation. The discrete frequency appears to increase (more oscillations) at first but then after three increments (half of the period, N) starts to back down again. Reaching the next harmonic at  $2\pi$ , the discrete signal is back to where it started. Further increases repeat the same cycle.

Let us take a closer look. The first row in Fig. 3.31 shows a zero-frequency harmonic. All real samples are 1.0, since this is a cosine. In (b), the continuous signal is of frequency 1 Hz, the discrete samples come from  $\cos(2\pi/6)$ . In (c), we see a continuous signal of 2 Hz and discrete samples from  $\cos(2\pi/3)$ . It is seen that by changing the phase from (a) to (g) we have gone through a complete  $2\pi$  cycle. In (g) the samples are identical to case (a) yet, continuous frequency is much higher. The samples for case (g) look exactly the same as for case (a) and case (h) looks exactly the same as case (b) and so on. These intermediate 6 harmonic in a  $2\pi$  range are distinct where those obtained by increasing the frequency by  $2\pi$  are not!

These harmonics are an orthogonal basis set and can be used to create a Fourier series representation of a discrete signal. The weighted sum of these N special signals forms the discrete Fourier series (DFS) representation of the signal. Unlike the CT signal, here the meaningful range of the harmonic signal is limited to a finite number of harmonics, which is equal to the period of the discrete signal in samples. Hence, the number of unique coefficients is finite and equal to the period N.

# **Discrete-Time Fourier Series Representation**

The **Discrete-time Fourier series (DTFS)** is the discrete representation of a DT periodic signal by a linear weighted combination of these  $K_0$  distinct complex exponentials.



Figure 3.21: The real and the imaginary component of the discrete signal harmonics. They are all different.

These distinct orthogonal exponentials exist within just one cycle of the signal with cycle defined as a  $2\pi$  phase shift. As the number of harmonics available is discrete, the spectrum is also discrete, just as it is for a continuous signal. We write the Fourier representation of the discrete signal x[n] as the weighted sum of these harmonics.

$$x[n] = \frac{1}{N_0} \sum_{k=0}^{K_0 - 1} C_k e^{jk\frac{2\pi}{N_0}n}$$
(3.36)

Equation (3.36) is the Fourier series representation of a discrete periodic signal.  $C_k$  are the complex Fourier series coefficients (FSC), which we discussed in Chapter 2. There are *k* harmonics, hence you see the coefficients with index *k*. In the exponent to the CE, the term  $\frac{2\pi}{N_0}$  is incremental frequency change, or the fundamental digital frequency of the signal. Term *k* increments this fundamental frequency, with *n* is the index of time for the DT signal.

The DTFS coefficients of the *k*th harmonic are exactly the same as the coefficient for a harmonic that is an integer multiple of  $N_0$  samples so that:

$$C_k = C_{k+mN_0}$$

The Inverse DTFSC equation is given by the *k*th coefficient and hence, the *k*th coefficient is given by

$$C_k = \frac{1}{N_0} \sum_{n=0}^{N_0 - 1} x[n] e^{-j\Omega_0 nk}$$
(3.37)

The  $(k + mK_0)$ th coefficient is given by

$$C_{k+mK_0} = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j(k+mN_0)\Omega_0 n}$$
$$= \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-jk\Omega_0 n} e^{-jmN_0\Omega_0 n}$$

The second part of the signal is equal to 1.

$$e^{-jmN_0\Omega_0 n} = e^{-jm2\pi n} = 1$$

(Because the value of the complex exponential at integer multiples of  $2\pi$  is equal to 1.0). Therefore, we have:

$$C_{k+mN_0} = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j(k+mK_0)\Omega_0 n} = C_k$$
(3.38)

This says that the harmonics repeat, and hence the coefficients also repeat. The spectrum of the discrete signal (comprising the coefficients) keeps repeating after every integer multiple of the  $K_0$  samples, or by the sampling frequency. In practical sense, this means we can limit the computation to just the first  $K_0$  harmonics, where,  $K_0$  is equal to  $N_0$ .

This is a very different situation from the continuous signals, which do not have such behavior. The CT coefficients are unique for all values of k.

#### **DTFS Examples**

Example 3.8. Find the DT Fourier series coefficients of this signal.

$$x[k] = 1 + \sin\left(\frac{2\pi}{10}k\right)$$

The fundamental period of this signal is equal to 10 samples from observation. Hence, it can only have at most 10 unique coefficients.



Figure 3.22: Signal of Ex. 3.8 and its Fourier coefficients (a) The discrete signal with period = 10, (b) the fundamental spectrum, (c) the true repeating spectrum.

Now we write the signal in complex exponential form.

$$x[n] = 1 + \frac{1}{2j}e^{j\frac{2\pi}{10}n} - \frac{1}{2j}e^{-j\frac{2\pi}{10}n}$$

Note that because the signal has only three components, corresponding to index k = -1, 0, and 1, for zero frequency and  $k = \pm 1$ , which corresponds to the fundamental

frequency, the coefficients for remaining harmonics are zero. We can write the coefficients as:

$$C_{k} = \frac{1}{N_{0}} \sum_{n=0}^{N_{0}-1} x[n] e^{-jk\Omega_{0}n}$$
  
=  $\frac{1}{10} \sum_{n=0}^{9} x[n] e^{-jk\frac{2\pi}{10}n}$   
$$C_{0} = \frac{1}{10} \sum_{n=0}^{9} x[n] e^{-j(k=0)\frac{2\pi}{10}n}$$
  
=  $\frac{1}{10} \sum_{n=0}^{9} x[n]$   
= 1

In computing the next coefficient, the value of the complex exponential for k = 1, and then for each value of k, we use the corresponding x[n] and the value of the complex exponential. The summation will give us these values.

$$C_{1} = \frac{1}{10} \sum_{n=0}^{9} x[n] e^{-j(k=1)\frac{2\pi}{10}n} = \frac{1}{2j}$$
$$C_{-1} = \frac{1}{10} \sum_{n=0}^{9} x[n] e^{-j(k=-1)\frac{2\pi}{10}n} = -\frac{1}{2j}$$

Of course, the coefficients can be seen directly in the complex exponential form of the signal. The rest of the coefficients from  $C_2$  to  $C_9$  are zero. However, the coefficients repeat after  $C_9$  so that  $C_{1+9k} = C_1$  for all k. This in shown in the spectrum of the signal in Fig. 3.23(c). In Figure 3.23(b), only the fundamental spectrum is shown, but infact the spectrum repeats every 10 samples, forever.

Example 3.9. Compute the DTFSC of this discrete signal.

$$x[n] = \frac{5}{2} + 3\cos(\frac{2\pi}{5}n) - \frac{3}{2}\sin(\frac{2\pi}{4}n)$$

The period of the second term, cosine is 5 samples and the period of sine is 4 samples. Period of the whole signal is 20 samples because it is the least common multiple of 4 and 5. This signal repeats after every 20 samples. The fundamental frequency of this signal is

$$\Omega_0 = \frac{2\pi}{20} = \frac{\pi}{10}$$



Figure 3.23: Signal of example 3.9 (a) The discrete signal with period = 20, (b) the fundamental spectrum, (c) the true repeating spectrum.

We calculate the coefficients as

$$C_{k} = \frac{1}{N_{0}} \sum_{n=0}^{N_{0}-1} x[n] e^{-jk\Omega_{0}n}$$
$$= \frac{1}{20} \sum_{n=0}^{19} x[n] e^{jk\frac{\pi}{10}n}$$
$$\Rightarrow C_{0} = \frac{1}{20} \sum_{n=0}^{N_{0}-1} x[0]$$

The FSC of this signal repeat with a period of 20. Each harmonic exponential varies in digital frequency by  $2\pi/20$ . Based on this knowledge, it can be shown that the  $2\pi/5$  exponential falls at k = 4,  $2(\pi/10) = 2\pi/5$  and exponential  $2\pi/4$  falls at k = 5,  $5\pi/10 = \pi/2$ .

We can also write this signal as

$$x[n] = \frac{5}{2} + \left(\frac{3}{2}e^{\frac{2\pi}{5}n} - \frac{3}{2}e^{-\frac{2\pi}{5}n}\right) + j\left(\frac{3}{4}e^{\frac{2\pi}{4}n} - \frac{3}{4}e^{-\frac{2\pi}{4}n}\right)$$

From here, we see that the zero-frequency harmonic has a coefficient of 5/2. The frequencies  $\pm 2\pi/5$  and  $\pm 2\pi/4$  have coefficients of 3/2 and 3/4 as shown in Fig. 3.23(b).

**Example 3.10.** Compute the DTFSC of a periodic discrete signal that repeats with period = 4 and has two impulses of amplitude 2 and 1, as shown in Fig. 3.24(a).

The period of this signal is 4 samples as we can see and its fundamental frequency is



 $\Omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$ 

*Figure 3.24: Signal of Ex. 3.10. The discrete signal with period = 4 (b) the fundamental spectrum, (c) the true repeating spectrum.* 

We write expression for the DTFSC from Eq.(3.37).

$$C_k = \frac{1}{4} \sum_{n=0}^{3} x[n] e^{-jk\frac{\pi}{2}n}$$

Solving this summation in closed form is hard. In nearly all such problems, we need to know series summations or the equation has to be solved numerically. In this case, the relationship is unknown. We first express the complex exponential in its Euler form. As known already (from Chapter 2), the values of the complex exponential for argument  $\pi/4$  are 0 and 1 respectively for the cosine and sine. It can be written in a concise form as:

$$\cos\left(\frac{\pi}{2}\right) = 0$$
 and  $\sin\left(\frac{\pi}{2}\right) = 1$ .

We get for the above exponential

$$e^{-jn\frac{\pi}{2}k} = \left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right)^{nk} = (-j)^{nk}$$

Now substitute this into the DTFSC equation and calculate the coefficients, knowing there are only four harmonics in the signal because the number of harmonics are equal
to the fundamental.

$$C_{k=0} = \frac{1}{4}x[n]e^{-jk\frac{\pi}{2}n} = 0.75$$

$$C_{k=1} = \frac{1}{4}x[n]e^{-jk\frac{\pi}{2}n} = 0.56$$

$$C_{k=2} = \frac{1}{4}x[n]e^{-jk\frac{\pi}{2}n} = 0.25$$

$$C_{k=3} = \frac{1}{4}x[n]e^{-jk\frac{\pi}{2}n} = 0.56$$

These four values can be seen repeated in Fig. 3.24(c).



Figure 3.25: (a) The discrete signal with period = 4; (b) the fundamental spectrum; (c) the true repeating spectrum.

Example 3.11. Find the DTFSC of the following sequence.

$$x[n] = \{0, 1, 2, 3, 0, 1, 2, 3, \ldots\}$$

The fundamental period of this series is equal to 4 samples by observation. We will now use a compact form of the exponentials to write out the solution.

$$W_4 = e^{-j\frac{2\pi}{4}} = \underbrace{\cos\left(\frac{2\pi}{4}\right)}_0 - j\underbrace{\sin\left(\frac{2\pi}{4}\right)}_1 = -j$$

Note that this *W* is not a variable but a constant. Its value for these parameters is equal to -j. Now we write the coefficients as

$$C_k = \frac{1}{4} \sum_{n=0}^{3} x[n] W_4^{nk}, \quad k = 0, \pm 1, \pm 2, \dots$$

From here we get

$$C_{0} = \frac{1}{4} \sum_{n=0}^{3} x[n] W_{4}^{0 \cdot n} = 1.5$$

$$C_{1} = \frac{1}{4} \sum_{n=0}^{3} x[n] W_{4}^{1 \cdot n} = \sum_{k=0}^{3} x[n] (-j)^{n} = 0.612$$

$$C_{2} = \frac{1}{4} \sum_{n=0}^{3} x[n] W_{4}^{2 \cdot n} = \sum_{k=0}^{3} x[n] (-j)^{2n} = 0.5$$

$$C_{3} = \frac{1}{4} \sum_{n=0}^{3} x[n] W_{4}^{3 \cdot n} = \sum_{k=0}^{3} x[n] (-j)^{3n} = 0.612.$$

#### DTFSC of a repeating square pulse signal

**Example 3.12.** Find the DTFSC of a square pulse signal of width *L* samples and period *N*.

Once again, we examine the FSC of a square pulse signal. The signal is discrete now, which means that it has a certain number of non-zero samples along with the zeros for the rest, making up one period of DT data. We set the period of the pulse to L samples, which is the width of the pulse. The length of the period is N samples. The duty cycle is defined as the ratio of the pulse width, L and the period, N. The pulse is not centered at the origin in this case, as shown in Fig. 3.26. This has the effect of introducing a phase term, as we shall see in the result.

To compute the DTFSC, the following important property of geometric series is used.

$$\sum_{n=0}^{M-1} a^n = \frac{1-a^M}{1-a}, \quad |a| < 1.$$
(3.39)

The coefficients of this signal are given as:

$$C_{k} = \frac{1}{N_{0}} \sum_{n=0}^{L-1} x[n] e^{-j\frac{2\pi}{N_{0}}nk} + \underbrace{\frac{1}{N_{0}} \sum_{n=L}^{N-1} x[n] e^{-j\frac{2\pi}{N_{0}}nk}}_{= \frac{1}{N_{0}} \sum_{n=0}^{L-1} 1 \cdot e^{-j\frac{2\pi}{N_{0}}nk}$$
$$= \frac{1}{N_{0}} \sum_{n=0}^{L-1} \left( e^{-j\frac{2\pi}{N_{0}}n} \right)^{k}$$

Now, we use the geometric series in Eq. (3.39) by setting *a* to  $(e^{-jk\frac{2\pi}{N}})$ . We now have:

$$a^n = (e^{-jk\frac{2\pi}{N}})^n$$

Using this term in Eq. (3.39), we get the following expression for the coefficients:

$$C_k = \frac{1}{N_0} \frac{1 - e^{-jk\frac{2\pi}{N_0}L}}{1 - e^{-jk\frac{2\pi}{N_0}}}$$

Now, pull out a common term from the numerator to write it as:

$$(e^{-jkL\frac{2\pi}{N_0}\frac{1}{2}})(\underline{e^{jkL\frac{2\pi}{N_0}\frac{1}{2}}}-e^{-jkL\frac{2\pi}{N_0}\frac{1}{2}})$$

The underlined part is equal to  $2j \sin(Lk\frac{\pi}{N})$ . Similarly by pulling out this common term,  $e^{-jk\frac{2\pi}{N}\frac{1}{2}}$ , from the denominator, we get

$$(e^{-jk\frac{2\pi}{N_0}\frac{1}{2}})(e^{jk\frac{2\pi}{N_0}\frac{1}{2}}-e^{-jk\frac{2\pi}{N_0}\frac{1}{2}})$$

The underlined term here is similarly equal to  $2j \sin(k\frac{\pi}{N})$ . Note the missing parameter *L*. From these, we now write the coefficients as:

$$\frac{1}{N_0} \frac{1 - e^{-j\frac{2\pi}{N_0}Lk}}{1 - e^{-j\frac{2\pi}{N_0}k}} = \frac{1}{N_0} \frac{e^{-j\frac{\pi}{N_0}Lk} \left(e^{j\frac{\pi}{N_0}Lk} - e^{-j\frac{\pi}{N_0}Lk}\right)}{e^{-j\frac{\pi}{N_0}k} \left(e^{j\frac{\pi}{N_0}k} - e^{-j\frac{\pi}{N_0}k}\right)}$$

We manipulate this expression a bit more to get Eq. (3.40).

$$C_{k} = \frac{1}{N_{0}} \frac{\sin(kL\pi/N_{0})}{\sin(k\pi/N_{0})} \underline{e^{-jk\frac{\pi}{N_{0}}(L-1)}}$$
(3.40)

It does not look much simpler! However, if we look only at its magnitude (the front part) it is about as simple as we can get in DSP, which is to say not a lot. The DTFSC magnitude for a general square pulse signal of width *L* and period *N* samples is given by the front part and the phase by the underlined part. If you try to plot this function for k = 0, you will get a singularity, so for this point, we can compute the value of the function using the L'Hopital's rule, which gives the value of this function as L/N, or the average valve of the function over the period of *N* samples and certainly that makes sense from what we know of the  $a_0$  value of a Fourier series. It is the DC value.



Figure 3.26: (a) The discrete signal; (b) DT Fourier series (DTFSC) coefficients, (c) the true repeating spectrum.

In Chapter 2, we computed the spectrum of a square pulse signal. The spectrum was *nonperiodic* sinc function, whereas this spectrum is *periodic*, i.e. repeats, a consequence of discrete time domain. This is an important new development and worth understanding. This function is called the **Dirichlet function** and is a periodic form of the sinc function. In Matlab, the Dirichlet function is plotted as follows:

```
8
    DTFSC a square pulse train
1
  N = 10; % Period
2
    = -15:14
3
  n2 = -3*pi: .01: 3*pi'
    = 5; % Wi\dt h of the pulse
  L
5
  mag = abs(diric(2*pi*n/N, L)); % Discrete
6
  mag2 = abs(diric(2*pi*n2/N, L)); % Continuous function
  phase = exp(-1i * n * (L - 1) * pi/N)
  stem(n, mag); grid on;
9
  hold on;
10
  plot(n2, mag2, '-.b')
```

In Figure 3.27 the coefficients of various square pulse signals are plotted. These should be studied to develop an intuitive feel for what happens as the sampling rate increases as well the effect of the duty cycle, i.e. the width of the pulse vs. the period. Note that the spectrum repeat with the sampling frequency. This was not the case for CT signal. The Fig. 3.27 shows that as the pulses get wider, the response gets narrower. When the pulse width is equal to the period, hence it is all impulses, an impulse train for a response. This is a very important effect to know.



Figure 3.27: Spectrum of a periodic discrete square pulse signal. We see the DTFSC change as the duty cycle (the width of the pulse) of the square pulse increases relative to the period. In (a), all we have are single impulses and, hence, the response is a flat line. In (f), as the width gets larger, the DTFSC take on an impulse like shape. Note that the underlying sinc does not change.

#### Power spectrum

In all the examples in this chapter, we have been plotting either the magnitude or the amplitude spectrum. The Power spectrum is a different thing. As per the Parseval's theorem, it is defined as:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{k=-\infty}^{\infty} \left| X[\frac{F_s}{n}k] \right|^2$$
(3.41)

Similar to the idea from circuits, to obtain power, we square the time-domain sample amplitude to obtain the instantaneous power at that time instant. If the unit of the amplitude is voltage, the units become voltage-squared. If all such individual squaredamplitudes are summed, we get the total power in the signal (assuming unity resistance.) Alternately, we can sum the square of the coefficients of each harmonic. They have units of amplitude too. This also gives us units of voltage-squared and can be used to convert a DTFSC into a Power spectrum by plotting the quantity  $(C_k)^2$  as a function of the index, k.

# Matrix method for Computing FSC

DTFSC are computed in closed-form for homework problems only. For practical applications, Matlab and other software and hardware devices are used. We will now look at a matrix method of computing the DTFSC.

Let us first define this common term. It looks strange and confusing, but it is a very simple idea. We use it to separate out the constant terms.

$$W_{N_0} \triangleq e^{-j\frac{2\pi}{N_0}}.$$
 (3.42)

For a given  $N_0$ , the signal period in samples, this term W, also called the **Twiddle factor** is a *constant* and is given a shorthand notation to make the equation writing easier. Using this factor, the DFS and the inverse DFS (IDFS) can be written as:

DFS: 
$$X[k] = \sum_{n=0}^{N_0 - 1} x[n] W_{N_0}^{nk}$$
  
IDFS:  $x[n] = \frac{1}{N_0} \sum_{k=0}^{N_0 - 1} X[k] W_{N_0}^{-nk}$  (3.43)

In this form, the terms  $W_{N_0}^{nk}$  and  $W_{N_0}^{-nk}$  are same as:

$$W_{N_0}^{nk} \triangleq e^{jk\frac{2\pi}{N_0}n}$$
$$W_{N_0}^{-nk} \triangleq e^{-jk\frac{2\pi}{N_0}n}$$

These terms can be precomputed and stored to make computation quicker. They are the basic idea behind the FFT algorithm which speeds up computation. The DTFSC equation can be setup in matrix form using the Twiddle factor and writing it in terms of two variables, the index n and k. Now we write:

$$C_{k} = \frac{1}{N_{0}} x[n] \begin{vmatrix} W^{-0 \times 0} & W^{-1 \times 0} & W^{-2 \times 0} & W^{-3 \times 0} \\ W^{-0 \times 1} & W^{-1 \times 1} & W^{-2 \times 1} & W^{-3 \times 1} \\ W^{-0 \times 2} & W^{-1 \times 2} & W^{-2 \times 2} & W^{-3 \times 2} \\ W^{-0 \times 3} & W^{-1 \times 3} & W^{-2 \times 3} & W^{-3 \times 3} \end{vmatrix}$$
(3.44)

Here, we have assumed that  $N_0 = 4$ . Each column represents the harmonic index k and each row the time index, n. It takes 16 exponentiations, 16 multiplications, and 4 summations to solve this equation. We will come back to this matrix methodology again when we discuss DFT and FFT in Chapter 6.

# **Summary of Chapter 3**

In this chapter, we examined discrete signals, the requirements for sampling set by Shanon and Nyquist, and methods of reconstruction. Discrete signals can experience frequency ambiguity, as many analog frequencies can fit through the same samples. If not sampled at a rate higher then Nyquist rate, we get aliasing. The Fourier series representation of discrete signals uses discrete basis functions. If a discrete signal has a period of *N* samples, then it only has *N* discrete harmonics within a  $2\pi$  range. In this chapter, we developed the FSC for discrete signals and looked at the inverse process.

The terms we introduced in this chapter:

- Discrete signals Defined only at specific uniform time intervals.
- **Digital signals** A discrete signals the amplitude of which is constrained to certain values. A binary digital signal can only take on two values, 0 or 1.
- **Nyquist rate** Two times the maximum frequency in the signal.
- Digital frequency Measured in samples per radian.
- Aliasing Given a set of discrete samples, a frequency ambiguity exists, as infinite number of frequencies can pass through these samples. This effect is called aliasing.
- **Discrete time Fourier series coefficients** DTFSC which repeat are the spectrum of a discrete signal.
- 1. An ideal discrete signal is generated by sampling a continuous signal with an impulse train of desired sampling frequency. The time between the impulses is called the sample time, and its inverse is called the sampling frequency.
- 2. The sampling frequency of an analog signal should be greater than two times the highest frequency in the signal to accurately represent the signal.
- 3. The fundamental period of a discrete periodic signal, given by  $N_0$ , must be an integer number of samples for the signal to be periodic in a discrete sense.
- 4. The fundamental frequency (or digital frequency) of a discrete signal, given by  $\Omega_0$ , is equal to  $2\pi/N_0$ .
- 5. The period of the digital frequency is defined as any integer multiple of  $2\pi$ . Harmonic discrete frequencies that vary by integer multiple of  $2\pi$ , such as  $2\pi k$  and  $2\pi k + 2\pi n$ , are identical.
- 6. We increment digital frequency by itself or  $2\pi/N_0$ , and use the  $N_0$  sub-frequencies resulting as the basis set. These frequencies are harmonic and distinct.

- 7. There are only  $K_0$  harmonics available to represent a discrete signal. The number of available harmonics for the Fourier representation,  $K_0$ , is exactly equal to the fundamental period of the signal,  $N_0$ .
- 8. Beyond the  $2\pi$  range of harmonic frequencies, the DT Fourier series coefficients, (DTFSC) repeat because the harmonics themselves are identical.
- 9. In contrast, the CT signal coefficients are aperiodic and do not repeat. This is because all harmonics of a continuous signal are unique.
- 10. The discrete Fourier series is written as:

$$x[n] = \frac{1}{N_0} \sum_{k=0}^{K_0 - 1} C_k e^{jk\frac{2\pi}{N_0}n}$$

11. The coefficients of the DTFS are written as:

$$C_k = \sum_{n=0}^{N_0 - 1} x[n] e^{-j\Omega_0 nk}$$

- 12. The coefficients of a discrete periodic signal are discrete just as they are for the CT signals.
- 13. The coefficients around the zero frequency are called the principal fundamental alias or principal spectrum.
- 14. The spectrum of a discrete periodic signal, repeats with sampling frequency,  $F_s$ .
- 15. Sometimes, DTFSC coefficients can be solved using closed form solutions but in a majority of the cases, matrix methods are used to find the coefficients of a signal.
- 16. Matrix method is easy to set up but is computationally intensive. Fast matrix methods are used to speed up the calculations. One such method is called FFT or Fast Fourier Transform.

# Questions

- 1. Given this CT signal, being sampled at  $F_s = 25$  Hz, write its discrete form. Is the sampling frequency above the Nyquist rate?  $x(t) = \sin(2\pi t) + 0.5\cos(7\pi t)$
- 2. If the largest frequency in a signal is  $f_1$  and the lowest is  $f_2$ , then what is the minimum frequency at which this signal should be sampled to be consistent with the Shannon's theorem?
- 3. Why would a signal be sampled at a rate higher than two times its maximum frequency?
- 4. A Gaussian signal does not have a clear maximum frequency. What frequency do you choose for sampling such a signal?
- 5. The bandwidth of a square pulse signal is infinite. How do you choose a sampling frequency for such a signal? What can you do to reduce the bandwidth of this signal?

- 6. For following three signals, what sampling frequency should be used so that it meets the Nyquist rate.
  - (a)  $x = \cos(50t)$
  - (b)  $y = \sin(30\pi t)$
  - (c)  $z = \sin(31.4(t+2)) \cos(40\pi t)$
- 7. What is the digital frequency of a signal given by these samples:  $x = [1 \ 1 \ -1 \ 1]$ .
- 8. What is the fundamental period of a sinusoid  $\cos(\Omega n + \phi)$ , the digital frequency of which is given by:  $0.4\pi$ ,  $0.5\pi$ ,  $0.6\pi$ , and  $0.75\pi$ .
- 9. A discrete signal repeats after 37 samples. Is it periodic? A discrete signal repeats with digital frequency of  $2\pi/5$ , is it periodic?
- 10. Stock data is noted every 5 s. What is its sampling frequency?
- 11. Temperature is measured every 10 min during the day and every 15 min during the night. 108 samples are collected over one day. Can we compute Fourier series coefficients of this data?
- 12. A signal has a period of 8 samples. What is its fundamental digital frequency?
- 13. Why would we want to recreate a signal from its samples?
- 14. Are we able to transmit discrete data over an rf(analog) link?
- 15. We have a sequence of alternating 0's and 1's. What might its DFS coefficients look like?
- 16. If you have a CT signal x(t) = cos(5t) and are told to sample it at four times the fundamental frequency, at what rate would you sample this signal?
- 17. A CT signal is given by  $x(t) = \sin(5\pi t)$ , if we sample it at a sampling frequency of 20 samples per second, how would you write the discrete version of this signal?
- 18. A signal is sampled at the rate of 15 samples per second. What frequency is represented by the harmonic index k = 3 if the harmonics range from, k = -7 to +7.
- 19. Digital frequency is limited to a range of 0 to  $2\pi$ . Why?
- 20. What is the minimum number of harmonics needed to represent this CT signal;  $x(t) = \sin(4\pi t + \pi/5)$ , with  $F_s = 25$ .
- 21. A discrete signal repeats after every 12 samples. What is its digital frequency? What is its fundamental frequency? What is its period?
- 22. A signal consists of three sinusoids of periods, N = 7, 9, and 11 samples. What is the fundamental period of the signal?
- 23. Why is a signal recreated using sinc reconstruction considered ideal?
- 24. If  $x[n] = 2\cos[(2\pi/5)n]$ , what are its DFS coefficients?
- 25. If  $x[n] = 1 \sin[(2\pi/8)n]$ , what are its DFS coefficients?
- 26. What happens to the spectrum of a train of square pulses as the pulses get narrower?
- 27. Why does the spectrum of a periodic discrete signal repeat? The repetition occurs over what frequency?

**Chapter 4** 

# Continuous-Time Fourier Transform of Aperiodic and Periodic Signals



Harry Nyquist February 7, 1889 - April 4, 1976

Harry Nyquist, was a Swedish-born American electronic engineer who made important contributions to communication theory. He entered the University of North Dakota in 1912 and received B.S. and M.S. degrees in electrical engineering in 1914 and 1915, respectively. He received a Ph.D. in physics at Yale University in 1917. His early theoretical work on determining the bandwidth requirements for transmitting information laid the foundations for later advances by Claude Shannon, which led to the development of information theory. In particular, Nyquist determined that the number of independent pulses that could be put through a telegraph channel per unit time is limited to twice the bandwidth of the channel. This rule is essentially a dual of what is now known as the Nyquist-Shannon sampling theorem. – From Wikipedia

# **Applying Fourier Series to Aperiodic Signals**

In previous chapters, we discussed the Fourier series as it applies to the representation of continuous and discrete signals. We discussed the concept of harmonic sinusoids as basis functions, first the trigonometric version of sinusoids and then the complex exponentials as a more compact form for representing the basis signals. The analysis signal is "projected" on to these basis signals, and the "quantity" of each basis function is interpreted as spectral content, commonly known as the spectrum.

Fourier series discussions assume that the signal of interest is periodic. However, a majority of signals we encounter in signal processing are not periodic. Many that we think are periodic are not really so. Furthermore, we have many signals that are bunch of random bits with no pretense of *periodicity*. This is the real world of signals and Fourier series comes up short for these types of signals. This was, of course, noticed right away by the contemporaries of Fourier when he first published his ideas is 1822. The Fourier series is great for periodic signals but how about stand-alone nonperiodic, also called *aperiodic* signals like this one?



Figure 4.1: Can we compute the Fourier series coefficients of this aperiodic signal?

Taking some liberty with history, Fourier, we are sure, must have been quite disappointed receiving a very unenthusiastic response to his work upon first publishing it. He was denied membership into the French Academy, as the work was not considered rigorous enough. His friends and foes, who are now as famous as he is (Laplace, Lagrange etc.) objected to his overreaching original conclusion about the Fourier series that it can represent *any* signal. They correctly guessed that series representation would not work universally, such as for exponential signals as well as for signals that are not periodic. Baron Fourier, disappointed but not discouraged, came back 20 years later with something even better, the *Fourier transform*. (If you are having a little bit of difficulty understanding all this on first reading, this is forgivable. Even Fourier took 20 years to develop it.)

# Extending the period to infinity

In this chapter, we will look at the mathematical trick Fourier used to modify the Fourier series such that it could be applied to signals that are transient or are not strictly periodic. Take the signal in Fig. 4.1. Let us say that this little signal, as shown, has been collected and the data show no periodicity. Being engineers, we want to compute its spectrum

using Fourier analysis, even though we have been told that the signal must be periodic. What to do?

Well, we can pretend that the signal in Fig. 4.1 is actually a periodic signal, but we are only seeing one period, the length of which is longer than the length of the data at hand. We surmise that if the length of our signal is 4 s, then maybe the signal looks like the top row of this figure with signal repeating with a period of 5 s. Of course, this is arbitrary. We have no idea what the period of this signal is, or if it even has one.



Figure 4.2: Going from periodic to aperiodic signal by extending the period.

Because a period of 5 seconds is an arbitrary number anyway, let us just increase it some more by pushing these assumed copies out, increasing the time in-between. We can indeed keep doing this such that the time goes on forever on each side and effectively the period becomes infinitely long. The signal is now just by itself with zeros extending to infinity on each side. We declare that this is now a periodic signal but with a period extending to  $\infty$ . We have turned an aperiodic signal into a periodic signal with this assumption.

We can apply the Fourier analysis to this extended signal because it is ostensibly *periodic*. Mathematically, we have let the period T go to infinity so that the assumed periods of the little signal move so far apart that we see neither hide nor hair of them. The single piece of the signal is then *one period* of a periodic signal, the other periods of which we can not see. With this assumption, the signal becomes periodic in a mathematical sense, and its Fourier series coefficients (FSC) can be computed by setting its period to,  $T = \infty$ . This conceptual trickery is needed because a signal must be periodic for Fourier series representation to be valid.

### **Continuous-Time Fourier Transform**

It was probably this same observation that led Fourier to the Fourier transform. We can indeed apply Fourier series analysis to an aperiodic signal by assuming that the period of an arbitrary aperiodic signal is *very* long and hence we are seeing only one period of the signal. The aperiodic data represents *one* period of a presumed periodic signal,  $\tilde{x}(t)$ . But if the period is infinitely long, then the *fundamental frequency* defined as the inverse of the period becomes infinitely small. The harmonics are still integer multiples of this infinitely small fundamental frequency but they are so very close to each other that they approach a continuous function of frequency. So a key result of this assumption is that the spectrum of an aperiodic signal becomes a continuous function of the frequency and is no longer discrete as are the FSC.

Figure 4.3(a) shows a pulse train with period  $T_0$ . The FSC of the pulse train are plotted next to it (See Ex. 2.10). Note that as the pulses move further apart in Fig. 4.3(b and c), the spectral lines or the harmonics are moving closer together.



Figure 4.3: Take the pulse train in (a), as we increase its period, i.e., allow more time between the pulses, the fundamental frequency gets smaller, which makes the spectral lines move closer together as in (c). In the limiting case, where the period goes to  $\infty$ , the spectrum would become continuous.

We will now go through the math to show how the **Fourier transform** (FT) is directly derived from the FSC. Like much of the math in this book, it is not complicated, only confusing. However, once you have clearly understood the concepts of fundamental frequency, period, and the harmonic frequencies, the rest gets easier.

After we discuss the continuous-time Fourier Transform (CTFT), we will then look at the discrete-time Fourier transform (DTFT) in Chapter 5. Of course, we are far more interested in a yet to be discussed transform, called the discrete Fourier transform (DFT). However, it is much easier to understand DFT if we start with the continuous-time case first. Although you will come across CTFT only in books and school, it is essential for the full understanding of this topic.

In Equation (4.1) the expression for the FSC of a continuous-time signal is repeated from Chapter 2.

$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt$$
(4.1)

To apply this to an aperiodic case, we let  $T_0$  go to  $\infty$ . In Eq. (4.1) as the period gets longer, we are faced with division by infinity. Putting the period in form of frequency avoids this problem. Thereafter, we only have to worry about multiplication by zero. We write the period as a function of the frequency.

$$\frac{1}{T_0} = \frac{\omega_0}{2\pi} \tag{4.2}$$

If  $T_0$  is allowed to go to infinity, then  $\omega_0$  is becoming tiny. In this case, we write frequency  $\omega_0$  as  $\Delta \omega$  instead, to show that it is becoming infinitesimally smaller. Now we write the period in the limit as

$$\lim_{T_0 \to \infty} \frac{1}{T_0} \approx \frac{\Delta \omega}{2\pi}$$
(4.3)

We rewrite Eq. (4.1) by substituting Eq. (4.3).

$$C_{k} = \frac{\Delta\omega}{2\pi} \int_{-T_{0}/2}^{T_{0}/2} x(t) e^{-jk\omega_{0}t} dt$$
(4.4)

But now as  $T_0$  goes to infinity,  $\Delta \omega$  approaches zero, and the whole expression goes to zero. To get around this problem, we start with the time-domain Fourier series representation of x(t), as given by

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\omega_0 t}$$
(4.5)

Now substitute Eq. (4.4) into Eq. (4.5) for the value of  $C_k$  to write

$$x(t) = \sum_{k=-\infty}^{\infty} \left\{ \underbrace{\frac{\Delta\omega}{2\pi} \int_{-T_0/2}^{T_0/2} x(t) e^{-j\omega t} dt}_{C_k} \right\} e^{jk\omega_0 t}$$
(4.6)

We now change the limits of the period,  $T_0$ , from a finite number to  $\infty$ . We also change  $\Delta \omega$  to  $d\omega$ , and  $k\omega_0$  to just  $\omega$ , the continuous frequency and the summation in Eq. (4.6) now becomes an integral. Furthermore, the factor  $1/2\pi$  is moved outside. Now, we

rewrite Eq. (4.6) incorporating these ideas as:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right\} e^{j\omega t} d\omega$$
(4.7)

We give the underlined part a special name, calling it the **Fourier transform** and refer to it by the expression,  $X(\omega)$ . Substituting this nomenclature in Eq. (4.7) for the underlined part, we write it in a new form. This expression is called the **inverse Fourier transform** and is equivalent to the Fourier series representation or the synthesis equation.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{X(\omega)e^{j\omega t} d\omega}{d\omega}$$
(4.8)

The CTFT is defined as the underlined part in Eq. (4.8) and is equal to

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$
(4.9)

In referring to the Fourier transform, the following terminology is often used. If x(t) is a time function, then its Fourier transform is written with a capital letter. Such as for time-domain signal, y(t) the CTFT would be written as  $Y(\omega)$ . These two terms are called a **transform pair** and often written with a bidirectional arrow in between them such as here.

$$\begin{array}{rcl} y(t) & \longleftrightarrow & Y(\omega) & y(t) & \stackrel{\vartheta}{\longleftrightarrow} & Y(\omega) \\ c(t) & \longleftrightarrow & C(\omega) & c(t) & \stackrel{\vartheta}{\longleftrightarrow} & C(\omega) \end{array}$$

The symbol  $\mathfrak{F}\{\cdot\}$  is also used to denote the Fourier transform. The symbol  $\mathfrak{F}^{-1}\{\cdot\}$  is used to denote the inverse transform such that

$$Y(\omega) = \mathfrak{F}\{y(t)\}$$
$$g(t) = \mathfrak{F}^{-1}\{G(\omega)\}$$

The CTFT is generally a complex function. We can plot the real and the imaginary parts of the transform, or we can compute and plot the magnitude, referred to as  $|X(\omega)|$  and the phase, referred to as  $\angle X(\omega)$ . The magnitude is computed by taking the square root of the product  $X(\omega)X^*(\omega)$  and phase by the arctan of the ratio of the imaginary and the real parts. We can also write the transform this way, separating out the magnitude and the phase spectrum.

$$X(\omega) = |X(\omega)|e^{j \angle X(\omega)}$$

Here we have the two important aspects of a spectrum, its magnitude and its phase.

- Magnitude Spectrum:  $|X(\omega)|$
- Phase Spectrum:  $\angle X(\omega)$ .

If the two components of a CE each have an amplitude 1.0 each, then its magnitude is equal to the square-root of 2. The phase of a CE is constant and equal to 0.

# **Comparing FSC and the Fourier Transform**



Figure 4.4: Fourier series and the Fourier transform

The Fourier series analysis can be used with both discrete and continuous-time signals as long as they are periodic. When a signal is aperiodic, the premium tool of analysis is the **Fourier transform**. Just as the Fourier series can be applied to continuous and discrete signals, the Fourier transform can also be applied to continuous and discrete signals. The discrete version of the Fourier transform is called the discrete-time Fourier transform, (DTFT) and will be discussed in the next chapter.

Let us compare the CTFT and the FSC equations. Recall that we are trying to determine the amplitudes of each of the harmonics used to represent the signal. The FSC and the CTFT are given as:

FSC: 
$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt$$
 (4.10)  
CTFT:  $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ 

In the CTFT expression, we note that the time no longer extends over a period, but extends to infinity. That is because the period itself now extends to infinity. We see that the period  $T_0$  in the front of FSC is missing from the latter. Where did it go and does it have any significance? We started development of CTFT by stretching the period and allowing it to go to infinity. We also equated  $1/T_0$  to  $d\omega/2\pi$  which was then associated with the time-domain formula or the inverse transform (it is not included in the center

part of Eq. (4.8), which became the Fourier transform). Therefore, it moved to the inverse transform as the factor  $2\pi$ .

Notice now the difference between the time-domain signal representation as given by the Fourier series and the Fourier transform.

FSC: 
$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$
  
CTFT:  $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ 
(4.11)

We no longer see the harmonic index k in the CTFT equation as compared to the FSC expression. This is because the frequency resolution  $\omega_0$  is infinitesimally small for CTFT and essentially the term  $k\omega_0$  is continuous. For example, if  $\omega_0$  is equal to 0.001, then integer multiple of this number are very close together and hence nearly continuous. The summation of the FSC over the harmonic index k hence, becomes an integral over  $\omega$  for the CTFT. Hence what is a summation for FSC, is an integral for the CTFT.

In Fourier series representation, to determine the quantity of a particular harmonic, the signal is multiplied by a harmonic, the product integrated over one period and result normalized by the fundamental period  $T_0$ . This gives the amplitude of that harmonic. Infact, this is done for all *K* harmonics, each divided by  $T_0$ . In Fourier transform, however, we do not divide by the period because we *don't know what it is*. It is assumed that it is  $\infty$ , but we would not want to divide by that either. Therefore, we just ignore it, and hence, we are not determining the signal's *true amplitude*. We are computing a measure of its content but it is not the actual content. Moreover, we are missing the same term from all coefficients, hence, the Fourier transform determines *relative amplitudes*. Very often, we are only interested in the relative levels of harmonic signal powers. The Fourier spectrum gives us the *relative distribution* of power among the various harmonic frequencies in the signal. In practice, we often normalize the maximum power to 0 dB such that the relative levels are consistent among all frequency components.

## **CTFT of Important Aperiodic Functions**

Now, we will take a look at some important aperiodic signals and their transforms, also called transform pairs. In the process, we will use the properties listed in Table 4.1. which can be used to compute the Fourier transform of many functions. These properties do not need proof as they are well known, and we will refer to them as needed for the following important examples. (In most cases, they are easy to prove.) The examples in this section cover some fundamental functions that come up both in workplace DSP as well in textbooks, so they are worth understanding and memorizing. We will use the

properties listed in Table 4.1 to compute the CTFTs in the subsequent examples in this and the following chapters. All following examples assume that the signal is aperiodic and is specified in continuous-time. The Fourier transform in these examples is referred to as CTFT.

Zero value	$X(0) = \int_{-\infty}^{\infty} x(t)  dt$
Duality	If $x(t) \leftrightarrow X(\omega)$ , then $X(t) \leftrightarrow x(\omega)$
Linearity	$ax(t) + by(t) \leftrightarrow aX(\omega) + bY(\omega)$
Time Shift	$x(t-t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$
Frequency Shift	$e^{j\omega_0 t}x(t) \leftrightarrow X(\omega - \omega_0)$
Time Reversal	$x(-t) \leftrightarrow X(-\omega)$
Time expansion or contraction	$x(at) \leftrightarrow \frac{1}{ a } X\left(\frac{\omega}{a}\right), a \neq 0.$
Derivative	$\frac{d}{dt}x(t) \longleftrightarrow j\omega X(\omega)$
Convolution in time	$x(t) * h(t) \leftrightarrow X(\omega)H(\omega)$
Multiplication in Time	$x(t)y(t) \leftrightarrow X(\omega) * Y(\omega)$
Power Theorem	$\int_{-\infty}^{\infty}  x(t) ^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty}  X(\omega) ^2 d\omega$

Table 4.1: Important CTFT properties

#### CTFT of an impulse function

Example 4.1.

$$x(t) = \delta(t) \tag{4.12}$$

This is the most important function in signal processing. The delta function can be considered a continuous (Dirac delta function) or a discrete function (Kronecker delta function), but here we treat it as a continuous function. We use the CTFT equation, Eq. (4.9) and substitute delta function for function x(t).



Figure 4.5: CTFT of a delta function located at time 0 is a constant.

We compute its CTFT as follows.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$
$$= \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} dt$$
$$= e^{-j\omega(t=0)}$$
$$= 1$$

In the third step, the sifting property of the delta function is used. The sifting property states that the integral of the product of a CT signal with a delta function isolates the value of the signal at the location of the delta function per Eq. (4.13).

$$\int_{-\infty}^{\infty} \delta(t-a)x(t) dt = x(a).$$
(4.13)

If a = 0, then the isolated value of the complex exponential is 1.0, at the origin. The integrand becomes a constant, so it is no longer a function of frequency. Hence, CTFT is constant for all frequencies. We get a flat line for the spectrum of the delta function.

The delta function was defined by Dirac as a summation of an infinite number of exponentials.

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega$$
(4.14)

The same equation in frequency domain is given by:

$$\delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} dt$$
(4.15)

The general version of the Dirac delta function with a shift for time and frequency is given as:

$$\delta(t-t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(t-t_0)} d\omega$$
(4.16)

$$\delta(\omega - \omega_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j(\omega - \omega_0)t} dt$$
(4.17)

In the transform of a delta function, we see a spectrum that encompasses whole of the frequency space to infinity, hence, a flat line from  $-\infty$  to  $+\infty$ . Infact when in Chapter 1, Fig. 1.9, we added a whole bunch of sinusoids, this is just what we were trying to get at. This is a very important property to know and understand. It encompasses much depth and if you understand it, the whole of signal processing becomes easier.

Now, if the CTFT or  $X(\omega) = 1$ , then what is the inverse of this CTFT? We want to find the time-domain function that produced this function in frequency domain. It ought to be a delta function but let us see if we get that. Using the inverse CTFT Eq. (4.8), we write

$$\begin{aligned} x(t) &= \mathfrak{F}^{-1}\{1\} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{j\omega t} \, d\omega \\ &= \delta(t) \end{aligned}$$

Substituting in the second step, the definition of the delta function from Eq. (4.14), we get the function back, a perfect round trip. The CTFT of a delta function is 1 in frequency-domain, and the inverse CTFT of 1 in frequency-domain is the delta function in time-domain.

$$\delta(t) \xrightarrow{\text{CTFT}} 1 \xrightarrow{\text{Inverse CTFT}} \delta(t)$$

#### CTFT of a constant

**Example 4.2.** What is the Fourier transform of the time-domain signal, x(t) = 1.

This case is different from Ex. 4.1. Here, the time-domain signal is a constant and not a delta function. It continues forever in time and is not limited to one single time instant as is the first case of a single delta function at the origin. Using Eq. (4.9), we write the CTFT as:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$
$$= \int_{-\infty}^{\infty} 1 \cdot e^{-j\omega t} dt$$

Using Eq. (4.15) for the expression of the delta function, we get the CTFT of the constant 1 as:

$$X(\omega) = 2\pi\delta(\omega)$$

It can be a little confusing as to why there is a  $2\pi$  factor, but it is coming from the definition of the delta function, per Eq. (4.15).



Figure 4.6: CTFT of a constant function that shows reciprocal relationship with Ex. 4.1.

If the time-domain signal is a constant, then its Fourier transform is the delta function and if we were to do the inverse transform of  $2\pi\delta(\omega)$ , we would get back x(t) = 1. We can write this pair as:

$$1 \leftrightarrow 2\pi \delta(\omega)$$

Note in Ex. 4.1, we had this pair  $\delta(t) \leftrightarrow 1$ , which is confusingly similar but is not the same thing. Note that in this case the FT is  $2\pi$  larger than the result from Ex. 1. In this case the signal continues forever so we ought to expect this FT to be larger.

### CTFT of a sinusoid

**Example 4.3.** Since a sinusoid is a periodic function, we will select only one period of it to make it *aperiodic*. Here, we have just a piece of a sinusoid. We make no assumption about what happens outside the selected time frame. The cosine wave shown in Fig. 4.7 has a frequency of 3 Hz, hence, you see one period of the signal lasting 0.33 s.



Figure 4.7: A piece of a sinusoid is an aperiodic signal.

We can compute the CTFT of this little piece of cosine as:

$$X(\omega) = \mathfrak{F}\{\cos(\omega_0 t)\}$$
$$= \int_{-\infty}^{\infty} \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t}) e^{j\omega t} dt$$
$$= \int_{-\infty}^{\infty} \frac{1}{2} e^{j(\omega + \omega_0)t} dt + \int_{-\infty}^{\infty} \frac{1}{2} e^{j(\omega - \omega_0)t} dt$$

Note that each of these integrals can be represented by a shifted delta function in frequency domain. We use Eq. (4.17) to write this result as:

$$X(\omega) = \pi \delta(\omega + \omega_0) + \pi \delta(\omega - \omega_0)$$
(4.18)

The only difference between the CTFT of a cosine wave and FSC in Ex. 4.1 is the scaling. In the case of FSC, we get two delta functions of amplitude 1/2 for each or a total of 1. The amplitude of each component in this case is, however,  $\pi$ , or  $2\pi$  times the amplitude of the FSC.

By similarity, the Fourier transform of a sine is given by

$$X(\omega) = \mathfrak{F}\{\sin(\omega_0 t)\} = j(\pi\delta(\omega + \omega_0) - \pi\delta(\omega - \omega_0))$$
(4.19)

The presence of *j* in front just means that this transform is in the imaginary plane. It has no effect on the amplitude. We do however, see that the component at  $-\omega_0$  frequency has a negative sign as compared to the cosine CTFT in Eq. (4.18) where both components are positive. This will come into play in the next example when we do the CTFT of a complex exponential.

#### CTFT of a complex exponential

**Example 4.4.** Now we calculate the CTFT of a very important function, the complex exponential.

$$x(t) = e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

A CE is really two functions, one a cosine of frequency  $\omega_0$  and the other a sine of the same frequency, both orthogonal to each other.

We have already calculated the CTFT of a sine and a cosine given by:

$$\mathfrak{F}\{\cos(\omega_0 t)\} = \pi \delta(\omega + \omega_0) + \pi \delta(\omega - \omega_0) \tag{4.20}$$

$$\mathfrak{F}\{\sin(\omega_0 t)\} = j\pi\delta(\omega + \omega_0) - j\pi\delta(\omega + \omega_0)) \tag{4.21}$$

By the linearity principle, we write the Fourier transform of the CE keeping the sine and cosine separate.



Figure 4.8: The CTFT of a complex exponential. The Real part is a cosine, hence the spectrum looks like Fig. 4.8(a) and the Imaginary part is a sine, and hence this plot is exactly the same as Fig. 4.8(b).

$$\mathfrak{F}\{e^{j\omega_0 t}\} = \mathfrak{F}\{\cos(\omega_0 t)\} + j\mathfrak{F}\{\sin(\omega_0 t)\}$$

$$= \underbrace{\pi \delta(\omega + \omega_0) + \pi \delta(\omega - \omega_0)}_{\cos} + \underbrace{j^2(\pi \delta(\omega + \omega_0) - j^2 \pi \delta(\omega - \omega_0))}_{\sin}$$

$$= \underbrace{\pi \delta(\omega - \omega_0) - j^2 \pi \delta(\omega - \omega_0)}_{add} + \underbrace{\pi \delta(\omega + \omega_0) + j^2 \pi \delta(\omega + \omega_0)}_{cancel}$$

$$= 2\pi \delta(\omega - \omega_0)$$
(4.22)

The result is a single delta function located at  $\omega_0$ . This happens because of an addition and a cancellation of the components. We see in Fig. 4.9 that the sine component at frequency  $+\omega$ , because it is multiplied by a j, rotates up or counter clockwise and adds to the cosine component. On the other side, at frequency  $-\omega$  the sine component coming out of the paper also rotates counter clockwise, because of its multiplication by j, which puts it directly in opposition to the cosine component and they both cancel. All we are left is a double component at the positive frequency. Hence, we get an asymmetrical result.

**Example 4.5.** Compute the iCTFT of a single impulse located at frequency  $\omega_1$ .

$$X(\omega) = \delta(\omega - \omega_1)$$

We want to know what time-domain function produced this spectrum. We take the iCTFT per Eq. (4.8).

$$\begin{aligned} x(t) &= \mathfrak{F}\{\delta(\omega - \omega_1)\} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_1) e^{j\omega t} \, d\omega = \frac{1}{2\pi} e^{j\omega_1 t} \end{aligned}$$

The result is a complex exponential of frequency  $\omega_1$  in time-domain. Because this is a complex signal, it has a non-symmetrical frequency response that consists of just one impulse located at the CE's frequency. In Fig. 4.9, we see why it is one sided.



Figure 4.9: The asymmetrical spectrum of a complex exponential.

Here, we write the two important CTFT pairs. The CTFT of a CE is one-sided, an impulse at its frequency. (The CTFT of all complex functions are asymmetrical.)

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$
 (4.23)

$$e^{-j\omega_0 t} \leftrightarrow 2\pi\delta(\omega+\omega_0)$$
 (4.24)

#### Time-shifting a function

Discrete signals can be constructed as summation of time-shifted delta functions. Hence, we ask, what is the CTFT of a delta function shifted by a time shift,  $t_0$ ? This case is very important to further understanding of discrete signals.

We can determine the response of a delayed signal by noting the time-shift property in Table 4.1. The property says that if a function is delayed by a time period of  $t_0$ , then in frequency domain, the original response of the undelayed signal is multiplied by a CE of frequency  $e^{j\omega t_0}$ . In this signal, time is constant and, hence, this is a frequency-domain signal, with frequency being the variable.

We write the shifted signal as  $x(t) = \delta(t - t_0)$ . Now we calculate the Fourier transform of this function from Eq. (4.9) as:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$
$$= \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t} dt$$
$$= e^{-j\omega t_0}$$

In Ex. 4.1, for a undelayed delta function, the CTFT was computed as 1.0. Here the result is the exponential due to the delay, hence proving the time-shift property. The CTFT of a delayed delta function is a CE. This CE has the form  $e^{-j\omega t_0}$  and might be confusing. That is because we are not used to seeing exponentials in frequency domain.

This is a really simple case so we may ask, what is the effect of delay on an arbitrary CTFT? Delaying a signal does not change its amplitude (the main parameter by which we characterize signals.) Its frequency also does not change, but what does change is its phase. If a sine wave is running and we arrive to look at it at time  $t_0$  after it has started, we are going to see an instantaneous phase at that time that will be different depending on when we arrive on the scene. That is all a time shift does. It changes the observed phase (or the starting point of the wave) of the signal.



Figure 4.10: Signal delay causes only the phase response to change. In (a) and (e) we see spectrum of an arbitrary undelayed signal, (b) the signal delayed by 2 s, (f) same signal delayed by 4 s. Both cases have the same magnitude as in (c) and (g) but the phase is different as in (d) and (h).

Figure 4.10 shows the effect of time-delay. In Fig. 4.10(a), a signal with an arbitrary spectrum centered at frequency of 2 Hz is shown. The time-domain signal is not shown, only its Fourier transform. You only need to note its shape and center location on the frequency axis. Now we delay this signal by 2 s (we do not know what the signal is, but that does not matter.) and want to see what happens to the spectrum.

In Fig. 4.10(b) we draw the CE  $e^{j\omega t_0}$  with  $t_0 = 2$  (both sine and cosine are shown). In Fig. 4.10(c), we see the effect of multiplying this CE by the spectrum in Fig. 4.10(a). The magnitude is unchanged. But when we look at Fig. 4.10(d) we see the phase. Since we do not know what the previous phase was, no statement can be made about it yet. Now examine the second column. In this case, the signal is delayed by 4 seconds. Once again in Fig. 4.10(g) we see no change in magnitude but we see that phase in Fig. 4.10(h) has

indeed changed from previous case in Fig. 4.10(d). The phase change is directly related to the delay.

#### Duality with frequency shift

If a signal is shifted in time, the response changes for phase but not for frequency. Now what if we shift the spectrum by a certain frequency, such as shifting  $X(\omega)$  to  $X(\omega - \omega_0)$ , i.e, the response is to be shifted by a constant frequency shift of  $\omega_0$ . We can do this by using the frequency shift property. In order to effect this change or frequency-shift, we need to change the time-domain signal as:

$$X(\omega - \omega_0) \leftrightarrow e^{j\omega_0 t} x(t) \tag{4.25}$$

Hence, if a time-domain signal is multiplied by a CE of a certain frequency, the result is a shifted frequency response by this frequency. We show this here.

$$\mathfrak{F}\{e^{j\omega_0 t}x(t)\} = \int_{-\infty}^{\infty} e^{j\omega_0 t}x(t)e^{-j\omega t} dt$$
$$= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t} dt$$
$$= X(\omega-\omega_0).$$
(4.26)

The frequency shift property shown here is also called the **modulation** property. Modulation, also called upconversion, can be thought of as multiplying, in time-domain, a signal by another single-frequency signal (called a *carrier*) and infact if you look at Eq. (4.25), that is exactly what we are doing. A time-domain signal multiplied by a CE,  $e^{j\omega_0 t}$  results in the signal transferring to the frequency of the CE,  $\omega_0$  without change in its amplitude.

#### **Convolution property**

The most important result from the Fourier transform is the convolution property. Infact the Fourier transform is often used to perform convolution in hardware instead of doing convolution in time-domain. The convolution property is given by

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau) d\tau \qquad (4.27)$$

In time-domain, convolution is a resource-heavy computation. However, convolution can be done using less computational effort using the Fourier transform. The convolution property states:

$$x(t) * h(t) \leftrightarrow X(\omega) H(\omega) \tag{4.28}$$

This states that the convolution of two signals can be computed by multiplying their individual Fourier transforms and then taking the inverse transform of the product. Let us see why this is possible. We write the time-domain expression for the convolution and then take its Fourier transform. Yes, it does look messy and requires fancy calculus.

$$\mathfrak{F}\{x(t) * h(t)\} = \mathfrak{F}\left\{\int_{-\infty}^{\infty} x(\tau)h(t-\tau) d\tau\right\}$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau e^{-j\omega t} dt$$

Now, we interchange the order of integration to get this from

$$\mathfrak{F}\{x(t)*h(t)\} = \int_{-\infty}^{\infty} x(\tau) \left( \int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega t} dt \right) d\tau$$

We make a variable change by setting  $u = t - \tau$ , hence we get

$$\mathfrak{F}\{x(t) * h(t)\} = \int_{-\infty}^{\infty} x(\tau) \left( \int_{-\infty}^{\infty} h(u) e^{-j\omega(u+\tau)} du \right) d\tau$$

This can be written as:

$$\mathfrak{F}\{x(t)*h(t)\} = \int_{-\infty}^{\infty} x(\tau) \left( \int_{-\infty}^{\infty} h(u)e^{-j\omega u}e^{-j\omega \tau} du \right) d\tau$$

Now we move the  $e^{-j\omega\tau}$  term out of the inner integral because, it is not function of u, to get the desired result and complete the convolution property proof.

$$\mathfrak{F}\{x(t) * h(t)\} = \left(\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau} d\tau\right) \left(\int_{-\infty}^{\infty} h(u)e^{-j\omega u} du\right)$$
$$= X(\omega)H(\omega)$$

The duality property of the Fourier transform then implies that if we multiply two signals in time-domain, then the Fourier transform of their product would be equal to convolution of the two transforms.

$$x(t)h(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * H(\omega)$$
 (4.29)

This is an efficient way to compute convolution rather than the standard way we learn by reversing and multiplying the signals. Convolution is hard to visualize. The one way to think of it is as smearing or a smoothing process. The convolution process produces the smoothed version of both of the signals as we can see in Fig. 4.11. Both of the pulses in Fig. 4.11(a and d) have been smoothed out by their convolution by the center pulse. They have also spread in time as in Fig. 4.11(c and f).



Figure 4.11: (a) The convolution of signals x(t) and y(t) in (c) is done using Fourier transform. In each case, the result is smoother than either of the original signals. Hence, convolution can be thought of as a filter.

#### CTFT of a Gaussian function

**Example 4.6.** Now we examine the CTFT of a really unique and useful function, the Gaussian. The zero-mean Gaussian function is given by

$$x(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/(2\sigma^2)}$$
(4.30)

where  $\sigma^2$  is the signal variance and  $\sigma$  the standard deviation of the signal. This form of the Gaussian is particular to signal processing. A general Gaussian signal is often written in this form.

$$f(t) = ae^{-(t-b)^2/2c^2}$$
(4.31)

Here a is the peak height of the familiar bell curve, b is the center point of the curve and c is the standard deviation.

The CTFT of the Gaussian function is very similar to the function itself.

$$X(\omega) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/(2\sigma^2)} e^{-j\omega t} dt$$
$$= \frac{1}{2\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2\sigma^2} e^{-j\omega t} dt$$

This is a difficult integral to solve but fortunately smart people have already done it for us. The result is

$$X(\omega) = \frac{1}{2\sigma\sqrt{2\pi}} e^{-\frac{\sigma^2 \omega^2}{2}}$$
(4.32)

As  $\sigma$  is a constant, the shape of this curve is a function of the square of the frequency, same as it is in time-domain where it is square of time. Hence, it is often said that the CTFT of the Gaussian function is same as itself, but what they really mean is that the shape is the same. This property of the Gaussian function is very important in nearly all fields. However, as we see in Eq. (4.32), there is no hard cutoff in the response and hence the bandwidth of this signal is not finite. How fast it decays depends on the parameter,  $\sigma$ , the standard deviation. If the x-axis is normalized by the standard deviation, the response becomes the normal distribution. In signal processing, we find that often noise can be modeled as a Gaussian process. We also find that when Gaussian signals are added or convolved, their joint distribution retains its Gaussian distribution.

#### CTFT of a square pulse

**Example 4.7.** Now, we examine the CTFT of a square pulse of amplitude 1, with a period of  $\tau$ , centered at time zero. This case is different from the ones in Chapters 2 and 3 in that here we have just a single solitary pulse. This is not a case of repeating square pulses as in this section we are considering only aperiodic signals.



Figure 4.12: Spectrum along the frequency line. A square pulse has a sinc-shaped spectrum. (a) its time-domain shape, and (b) its CTFT.

We write the CTFT as given by Eq. (4.9). The function has an amplitude of 1.0 for the duration of the pulse.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$
$$= \int_{-\tau/2}^{\tau/2} 1 \cdot e^{-j\omega t} dt$$
$$= -\frac{e^{-j\omega t}}{j\omega} \Big|_{-\tau/2}^{\tau/2}$$

$$= -\frac{1}{j\omega} \left( e^{-j\omega\tau/2} - e^{j\omega\tau/2} \right)$$
$$= \frac{2}{\omega} \sin\left(\frac{\omega\tau}{2}\right)$$

This can be simplified to

$$X(\omega) = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right) \tag{4.33}$$

The spectrum is shown on the right side in Fig. 4.12 for t = 1s and 2s. Note that as the pulse gets wider, the spectrum gets narrower. As the sinc function is zero for all values that are integer multiple of  $2\pi$ , the zero crossings occur whenever  $\omega \tau = k\pi$ , where k is an even integer larger than 2. For  $\tau = 2$ , the zeros would occur at radial frequency equal to  $\pi, 2\pi, \ldots$ . If the pulse were to become infinitely wide, the CTFT would become an impulse function. If they were infinitely narrow as in Ex. 4.1, the frequency spectrum would be flat.

Now assume that instead of the time-domain square pulse shown in Fig. 4.12, we are given a frequency response that looks like a square pulse. The spectrum is flat from -W to +W Hz. This can be imagined as the frequency response of an ideal filter. Notice, that in the time-pulse case, we defined the half-width of the pulse as  $\tau/2$ , but here we define the half bandwidth by W and not by W/2. The reason is that in time-domain, when a pulse is moved, its period is still  $\tau$ . However, bandwidth is designated as a positive quantity only. There is no such thing as a negative bandwidth. In this case, the bandwidth of the signal (because it is centered at 0 is said to be W Hz and not 2W Hz. However, if this signal was moved to a higher frequency such that the whole signal was in the positive frequency range, it would be said to have a bandwidth of 2W Hz. This crazy definition gives rise to the concepts of low-pass and band-pass bandwidths. Low-pass is defined as being centered at the origin so it has half the bandwidth of band-pass.



Figure 4.13: Time-domain signal corresponding to the rectangular frequency response.

**Example 4.8.** What time-domain signal produces a rectangular frequency response shown in Fig. 4.13(b)? The frequency response is limited to a certain bandwidth, W Hz.

$$X(\omega) = \begin{cases} 1 & |\omega| \le W \\ 0 & |\omega| > W \end{cases}$$

We compute the time-domain signal by the inverse CTFT equation.

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{j\omega t} \, d\omega \\ &= \frac{1}{2\pi} \frac{e^{j\omega t}}{jt} \Big|_{-W}^{W} \end{aligned}$$

Which can be simplified to

$$x(t) = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wt}{\pi}\right)$$
(4.34)

Again we get a sinc function, but now in time-domain. This is the duality principle at work. This is a very interesting case and of fundamental importance in communications.

The frequency spectrum shown in Fig. 4.13(b) is, of course, a very desirable frequency response. We want the frequency response to be tightly constrained. The way to get this type of spectrum is to use a time-domain signal that has a sinc pulses. But a sinc function looks strange as a time-domain signal because it is of infinite length. However, because it is "well-behaved," which means it crosses zeros at predicable points, we can use it as a signal shape, at least in theory. In practice, it is impossible to build a signal shape of infinite time duration. It has to be truncated, however, truncation causes distortion and we do not get the perfect brick-wall frequency response. An alternate shape with similar properties is the **raised cosine**, most commonly used signal shape in communications. The raised cosine shape is also truncated to shorten its length but its distortion is manageable because it decays faster than a sinc shape.

In Fig. 4.14 some important Fourier transforms of aperiodic signals are given. A good engineer should know all of these by heart.

## Fourier Transform of Periodic Signals

The Fourier transform came about so that the Fourier series could be made rigorously applicable to aperiodic signals. The signals we examined so far in this chapter are all *aperiodic*, even the cosine wave, which we limited to one period. Can we use the CTFT for a *periodic* signal? Our intuition says that this should be the same as the Fourier series. Let us see if that is the case.

Take a periodic signal x(t) with fundamental frequency of  $\omega_0 = 2\pi/T_0$  and write its FS representation.

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\omega_0 t}$$



Figure 4.14: Response to a Sinc shaped time domain signal.

Taking the CTFT of both sides of this equation, we get

$$X(\omega) = \mathfrak{F}\{x(t)\} = \mathfrak{F}\left\{\sum_{k=-\infty}^{\infty} C_k e^{j\omega_0 t}\right\}$$

We can move the coefficients out of the CTFT because they are not function of frequency. They are just numbers.

$$X(\omega) = \sum_{k=-\infty}^{\infty} C_k \,\mathfrak{F}\{\underline{e^{j\omega_0 t}}\}\$$

The Fourier transform of the complex exponential  $e^{j\omega_0 t}$  is a delta function located at the frequency  $\omega_0$  as in Ex. 4.4. Making the substitution, we get

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega - k\omega_0)$$
(4.35)

This equation says that the CTFT of a periodic signal is a *sampled* version of the FSC. The FSC are being sampled at frequency of the signal,  $\omega_0$ , with *k* the index of repetition. However, the FSC are already discrete! Thus, the only thing the Fourier transform does is change the scale. The magnitude of the CTFT of a periodic signal is  $2\pi$  times bigger than that computed with FSC as seen by the factor  $2\pi$  front of Eq. (4.35).

Important observation: The CTFT of an aperiodic signal is aperiodic and continuous whereas the CTFT of a periodic signal is aperiodic but discrete.

#### CTFT of a periodic square pulse train

**Example 4.9.** Now we examine the CTFT of the periodic square pulse. For the Fourier transform of this periodic signal, we will use Eq. (4.35)

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega - k\omega_0)$$

The FSC of a periodic pulse train with duty cycle = 1/2 are computed in Chapter 2 and given as

$$C_k = \frac{1}{2}\operatorname{sinc}(k\pi/2)$$

We plot these FSC in Fig. 4.15(b). To compute CTFT, we set  $\omega_0 = 1$  and now we write the CTFT expression as:

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega - k(\omega_0 = 1))$$

The result is the sampled version of the FSC scaled by  $2\pi$  (See Ex. 2.10) which are of course themselves discrete.



Figure 4.15: The periodic square wave with duty cycle of 0.5. (b) Its FSC and (c) its CTFT. Only the scale is different.

What if the square pulse was not centered at 0 but shifted some amount. We can compute the CTFT of this periodic function by applying the time-shift property to the CTFT of the unshifted square wave.



Figure 4.16: A time-shifted square pulse train.

This periodic function is same as Fig. 4.15 but is time-shifted. We can write it as:

$$y(t) = x(t - \tau/2)$$

By the time-shift property, we can write the CTFT of this signal by multiplying the CTFT of the unshifted case by  $e^{j\omega\tau/2}$ . Hence

$$Y(\omega) = X(\omega)e^{j\omega\tau/2}$$

which is

$$Y(\omega) = \left(2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega-k)\right) e^{j\omega\tau/2}$$

This time shift has no effect on the shape of the response at all, just as we would expect. Only the phase gets effected by the time shift.

The main reason we do a Fourier transform rather than the Fourier series representation and its coefficients, is that the Fourier transform can be used for *aperiodic* and *periodic* signals. There is, however, a key difference between the FSC and the CTFT. We calculate actual amplitudes with FSC. However, in developing the Fourier transform, we dropped the concept of a period, hence, the results are useful in a relative sense only. The Fourier transform is not a tool for measuring the real signal amplitudes but is mostly used as a qualitative tool for assessing relative amplitudes, power and issues of bandwidth occupation.

# **Summary of Chapter 4**

In this Chapter we looked at aperiodic signals and their frequency representations. The FS concept is extended so that Fourier analysis can be applied to aperiodic signals. In a manner similar to computing the coefficients, we call the process of computing the coefficients of aperiodic signal the Fourier transform. The spectrum of continuous signals using the Fourier transform is continuous, where the Fourier transform of a periodic signal is discrete.

Terms used in this chapter:

- Fourier Transform, FT
- Continuous-time Fourier Transform, CTFT
- Discrete-time Fourier Transform, DTFT
- **Transform pair** The signal in one domain and its Fourier transform in the other domain are called a Fourier transform pair.
- 1. Aperiodic signals do not have mathematically valid FSC.
- 2. Fourier transform is developed by assuming that a aperiodic signal is actually periodic but with an infinitely long period.
- 3. Whereas the spectrum of a periodic signal is represented by the Fourier series coefficients, the spectrum of an aperiodic signal is called the Fourier transform.
- 4. The term Fourier transform applies not just to aperiodic signals but to periodic signals as well.
- 5. The FSC are discrete whereas the CTFT of an aperiodic signal is continuous in frequency.

- 6. The CTFT of aperiodic signals is aperiodic.
- 7. The CTFT and the iCTFT are computed by:

CTFT 
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$
  
iCTFT  $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$ 

8. A function and its Fourier transform are called a transform pair.

$$x(t) \leftrightarrow X(\omega)$$

9. The CTFT of a periodic signal is given by the expression

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega - k\omega_0)$$

- 10. The CTFT of a periodic signal can be considered a sampled version of the FSC.
- 11. Unlike the CTFT of an aperiodic signal, the CTFT of a periodic signal is discrete, just as are the FSC for a periodic signals.

#### Questions

- 1. What is the conceptual difference between the Fourier series and the Fourier transform?
- 2. Why is the CTFT continuous? Why are the CTFSC coefficients discrete?
- 3. What is the CTFT magnitude of these impulse functions:  $\delta(t-1)$ ,  $\delta(t-2)$ ,  $\delta(t-T)$ .
- 4. Give the expression for the CTFT of a cosine and a sine. What is the difference between the two?
- 5. Given a sinusoid of frequency 5 Hz. What does its CTFT look like?
- 6. What is the difference between the Fourier transform magnitude of a sine and a cosine of equal amplitudes?
- 7. What is the CTFT (amplitude) of these sinusoids:  $sin(-800\pi t)$ ,  $-cos(250\pi t)$ ,  $0.25 sin(25\pi t)$ . What is the magnitude spectrum of these sinusoids?
- 8. What is the value of  $\sin(500\pi t)\delta(t)$ ,  $\cos(100\pi t)\delta(t-\pi)$ ?
- 9. What is the CTFT of:  $x(t) = 6\sin(10\pi t) 4\cos(4\pi t)$ ?
- 10. If the FT of a signal is being multiplied by this CE:  $e^{-j6\pi f}$ , what is the resultant effect in time domain?
- 11. We multiply a signal in time domain by this CE:  $e^{-j12\pi t}$ , what is the effect in frequency domain on the FT of the signal?
- 12. The summation of complex exponentials represents what function?
- 13. What is the value of  $\cos(6\pi t)\delta(t-4)$ ?
- 14. What is the CTFT of the constant  $\pi$ ?
- 15. A sinc function crosses first zero at  $\pi/B$ . What is its time domain equation? What does the spectrum look like and what is its bandwidth?
- 16. A sinc function crosses first zero at t = 1, give its time domain equation? What does the spectrum look like and what is its bandwidth?
- 17. What is the CTFT of  $sin(5\pi t) * \delta(t-5)$ ?
- 18. A signal of frequency 4 Hz is delayed by 10 s. By what CE do you multiply the unshifted CTFT to get the CTFT of the shifted signal?
- 19. Given  $x(t) = \operatorname{sinc}(t\pi)$ , at what times does this function cross zeros?
- 20. The first zero-crossing of a sinc function occurs at time = B s; 0.5 s; 2 s. What is the bandwidth of each of these three cases?
- 21. What is the main lobe width of the CTFT of square pulses of widths:  $T_s$ ,  $T_s/2$ ,  $\pi/2$ , and 3 s.
- 22. If the main lobe width of a sinc function (one sided) is equal to  $\pi/2$ , then how wide is the square pulse in time?
- 23. What is the CTFT of an impulse train with period equal to 0.5 s. Is this a periodic signal?
- 24. Convolution in time domain of two sequences represents what in frequency domain?
- 25. What is the Fourier transform of an impulse of amplitude 2 V in time domain?
- 26. If a signal is shifted by 2 s, what happens to its CTFT?
- 27. The CTFT of a periodic signal is continuous while the CTFSC is discrete. True or false?
- 28. If the CTFSC of a signal at a particular harmonic is equal to 1/2, then what is the value obtained via CTFT at the same harmonic?
- 29. Given the FSC of a signal, how do you calculate the FT of this signal?
- 30. What is modulation and how is it accomplished?
- 31. Why do the base-band and pass-band bandwidths of a signal differ?