Chapter 4

Fourier Transform of continuous and discrete signals

In previous chapters we discussed Fourier series (FS) as it applies to the representation of continuous and discrete signals. It introduced us to the concept of complex exponential signals that can be used as basis functions. The signal is then "projected" on these basis signals, and the "quantity" of each basis function is interpreted as spectrum along a frequency line. The idea of spectrum has many names in the literature such as: gain, frequency response, rejection, magnitude, power spectrum, power spectral density etc. . They are all referring to this distribution of signal content over a certain frequency band.

Because the basis set for Fourier analysis is discrete, the spectrums computed are also discrete. Fourier series discussions however always assume that the signal under our microscope is periodic. But a majority of signals we encounter in signal processing are not periodic. Even those that we think are periodic, such as an EKG which looks periodic, are not really so. Each period is slightly different.

Fourier, I am sure was pretty excited when he first came up with the idea of using Fourier series for all kinds of signal analysis, but unfortunately some of his contemporary jumped up and objected to his overreaching conclusion. They correctly guessed that series representation would not work for many signals, such as those that go off into space, like the tan function, the growing exponential and many others, including ones that have too many discontinuities and as well as a large class of signals that are not periodic. Baron Fourier went home disappointed from his big meeting with the likes of Lagrange and Laplace, but came back 20 years later with something even better, called the Fourier transform. (So if you are having a little bit of difficulty understanding all this on first reading, this is forgivable. Even Fourier took 20 years to develop it.)

In this chapter we will look at the mathematical trick Fourier used to extend the analysis to aperiodic signals. Take the signal in Fig. 4.1(a). This is not a periodic signal. There is information only in a few samples in the middle. We want to compute its spectrum using Fourier analysis but we have been told that the signal must be periodic. What to do?

In order to compute the Fourier coefficients of such a signal, we can assume that it is repeating by creating what is called a periodic extension, $\tilde{x}_N[k]$ with a squiggle over x to indicate that this is an extended version of the signal. This is shown in Fig. 4-1(b). We assume that the information signal (samples 8 to 12) repeat every 8 samples, or with N = 8. Okay, now we can compute Fourier series coefficients (FSC) of this extended signal because it is periodic. But this is not the signal we started with. So let's just keep pushing these side copies out by increasing the space between the information samples. We can keep doing this, such that the zeros go on forever on each side and effectively the period becomes infinitely long. The signal now has just the information part with zeros extending to infinity on each side. We declare, this is now a periodic signal with N = ∞. We have turned an aperiodic signal into a periodic signal by this assumption. And indeed this is perfectly valid. We can now apply the FS analysis to this extended signal.

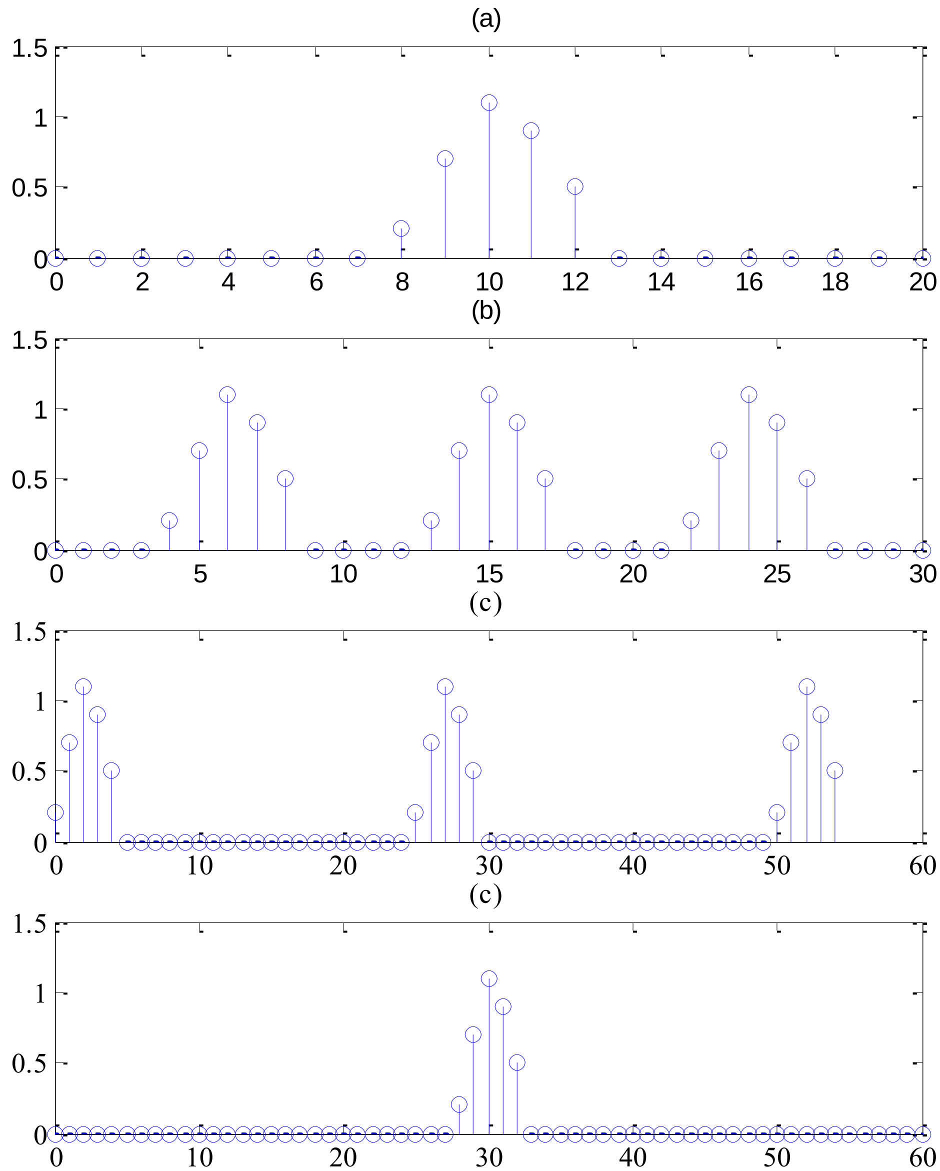


Figure 4.1 – Going from a periodic to a non-periodic signal

Recall that we write the FS for a continuous signal in terms of its complex coefficients as

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{-jn\omega_0 t} \tag{1.1}$$

And the coefficients C_n are given by

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt \qquad (1.2)$$

Here ω_0 is the fundamental frequency of the signal and n the index of the harmonic such that $n\omega_0$ is the nth harmonic. The period of the signal is called T for the continuous case as K_0 for the discrete case. In the discrete case, the sample number k, is also called the bin number. The frequency spac between the bin for the continuous case is ω_0 and $\frac{2\pi}{K_0}$ for the discrete case.

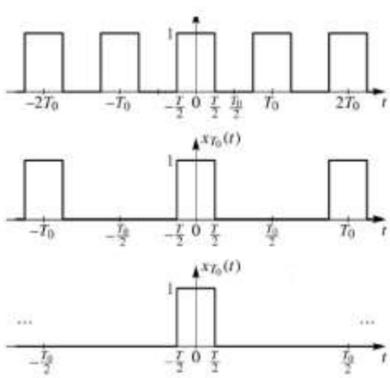
What happens to the coefficients of a periodic series as we stretch the period by adding more and more zeros in between the information pieces?

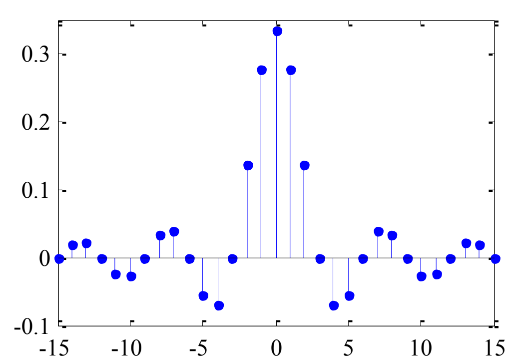
The frequency resolution becomes smaller and smaller as period increases. As we increase T, the fundamental frequency which is equal to $\omega_0 = 2\pi / T$, gets smaller, hence the space between the harmonics also becomes smaller. The consequence of T going to ∞, is that ω_0 approaches zero and the summation in Eq. (1.1) effectively becomes an integral, resolution becoming a continuous variable $\Delta\omega$.

In the limit, we can replace the discrete harmonics which are an integer multiple of the fundamental frequency with a continuous frequency, ω since they are now so close together that they are essentially continuous.

Take a look at signal in Fig. 4.2. In the first figure we show a pulse train and its CTFS in (a), (b) and (c) as we push the signal period out. Note that as the pulses move further apart, the harmonics begin to move closer together, i.e. there is more of them in each lobe.

The last lone pulse is the aperiodic signal and it is not hard to imagine looking at the way these harmonics are getting closer together that its FSC will become continuous.





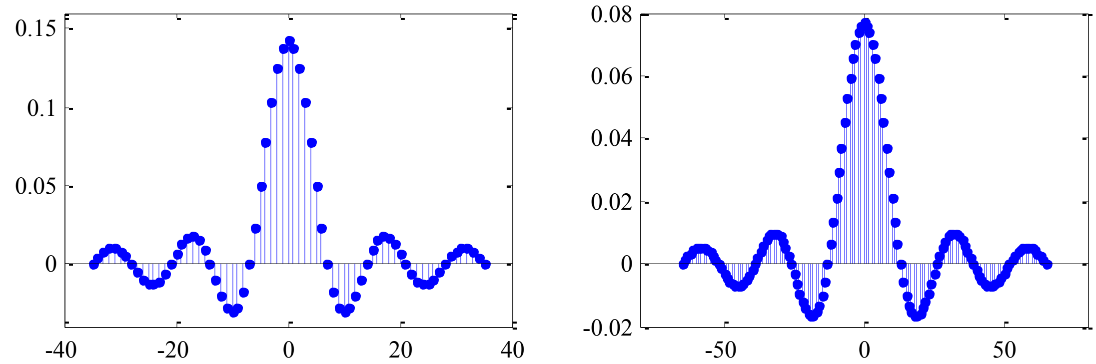


Figure 4.2 - Stretching the period, makes the fundamental frequency smaller, which makes the spectral lines move closer together.

Key idea: Increasing the period of a signal allows us to create an aperiodic version of the signal. The increasing period brings harmonics closer together, so that the spectrum of an aperiodic signal becomes continuous.

Continuous-time Fourier Transform (CTFT)

We can apply Fourier series analysis to a non-periodic signal and the spectrum will now have a continuous distribution instead of the discrete one we get for periodic signals. This idea of extending the period which results in this change is our segway into the concept of Fourier transform. We will now discuss how Fourier transform (FT) is derived from the Fourier series coefficients (FSC). After we discuss the continuous-time Fourier transform (CTFT), we will then look at the discrete-time Fourier transform (DTFT).

We write the Fourier series coefficients of a continuous-time signal once again as

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{j\omega_n t}\, dt \tag{1.3}$$

Where ω_n is the n_{th} harmonic or is equal to n times the fundamental frequency, $n\omega_0$, and T is the period of the fundamental frequency. In order to make T go to infinity, we make a couple of changes in the formula. First we substitute this into Eq. (1.3)

$$\frac{1}{T} = \frac{\omega_0}{2\pi} \tag{1.4}$$

Then we make ω_0 a function of the period T and write it as the infinitesimal $\Delta\omega$. We do that, because in Eq. (1.4) as period T, gets larger, we are faced with a division by infinity. Putting the period in form of frequency avoids this problem. Then we only have to worry about multiplication by zero! Now we add the limit in the front and change the limits of integration to length of the signal. Since the signal is zero outside of these limits, we do not need to go further.

$$C_n = \lim_{T\to\infty} \frac{\Delta\omega}{2\pi} \int_{-T/2}^{T/2} x(t)\, e^{-j\omega_n t}\, dt \tag{1.5}$$

This expression is not very helpful so far, because as T goes to infinity, $\Delta\omega$ goes to zero, so the whole expression goes to zero. But now we substitute this equation into the expression of the Fourier series itself. The expression foe the Fourier series is given by:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{j\omega_n t} \tag{1.6}$$

Now substitute into this equation, the value of C_n form Eq. (1.5) modified for an extended period case, we get

$$x(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \left\{ \frac{\Delta\omega}{2\pi} \int_{-T/2}^{T/2} x(t) e^{-j\omega_n t} dt \right\} e^{j\omega_n t} \tag{1.7}$$

Notice what happened here, we substituted into Eq. (1.6), the modified value of the extended period coefficients from Eq. (1.6). Now as T goes to infinity, range of integration in the middle integral changes again from $-\infty$ to $+\infty$. Also because the harmonics move so close to each other that we call them by just ω, a continuous variable, instead of ω_n. The summation on the outside also becomes an integration because we are now multiplying the coefficients (the middle part) with $\Delta\omega$, kind of like computing an infinitesimal area. Now we rearrange this combination of the two equations as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(t) e^{-j\omega_n t} dt \right\} e^{j\omega_n t} d\omega \tag{1.8}$$

Because the middle part is now a function of the continuous frequency, we give it a special name, calling it the **Fourier transform**. Notice that the formula outside of this term is that of the Fourier series.

$$X(\omega)=\int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt \tag{1.9}$$

This is the *formula for the coefficients of a non-periodic signal*. The time-domain signal is obtained by substituting $X(\omega)$ back into Eq. (1.8) as

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega t}\,d\omega \tag{1.10}$$

Summarizing we have the **Fourier transform** of a **continuous-time non-periodic** signal as

$$X(\omega)=\int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt \tag{1.11}$$

The formula for the time-domain signal, Eq. (1.10) is called the **Inverse Fourier Transform**.

In frequency form the two formulas are written as

Forward Fourier transform

$$X(f)=\int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft} \tag{1.12}$$

Inverse Fourier Transform

$$x(t)=\int_{-\infty}^{\infty} X(f)\,e^{j2\pi ft}df \tag{1.13}$$

In these formulas as compared to the Fourier series formula of Eq. (1.3), we no longer have the discrete harmonic index n to denote the nth harmonic because the frequency is now continuous.

Comparing Fourier series coefficients and Fourier transform

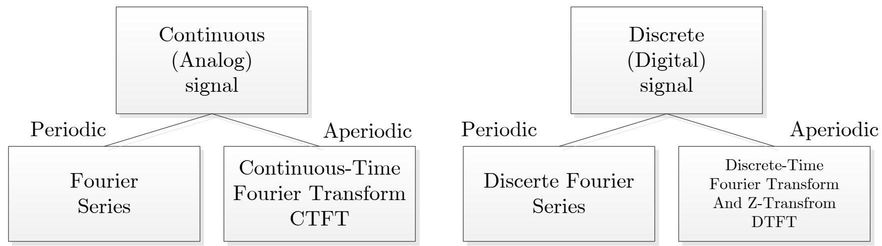


Figure 4.3 – Fourier series and Fourier Transform

The Fourier series is supposedly valid only for periodic signals. We can use Fourier sries analysis with both discrete and continuous-time signals as long as they are periodic. When the signal is non-periodic, the tool of analysis is the Fourier Transfrom. Just as Fourier series can be applied to continuous and discrete signals, the Fourier transfrom also has two forms, one for continuous and the other for a discete signal. The spectrum obtained from FS is discrete for both types of signals, but Fourier transfrom gives us a continuous spectrum instead.

Let's compare the Fourier transform (FT) with the Fourier series coefficient (FSC) formula for a continuous-time periodic signal. The FSC and the FT formulas are given as:

$$\begin{aligned} C_n &= \frac{1}{T}\int_0^T x(t)\, e^{-j\omega_n t} dt \qquad FSC \\ X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \qquad FT \end{aligned} \tag{1.14}$$

When we compare FSC with the FT formulas, we see that they are nearly the same except that the term 1/T in the front is missing from the latter. Where did it go and does it have any significance? We started development of FT by assuming that T goes to infinity, and then we equated 1/T to Δf and again mapped it to a continuous variable ω by turning it into $d\omega$. The $d\omega$ was then associated with the time-domain formula or the inverse transform (notice, it is not included in the center part of Eq. (1.8), which became the Fourier transform.). So it moved to the inverse transform in form of a 2π factor. The other difference is that the frequency ω is continuous for the FT.

Notice the difference between the time-domain signals as given by FS and FT.

$$
\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} C_n \, e^{j\omega_n t} \qquad FS \\
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega \quad FT
\end{aligned} \tag{1.15}
$$

In FS. to determine the quantity of a particular harmonic, we multiplied the signal by that harmonic, integrated the product over one period and divided the result by T. This gave us the amplitude of that harmonic. (See Chapter 1). In fact we do that for all harmonics, each divided by T. But here in the case of the Fourier transform, ostensibly we are doing the same thing but we do not divide by period T. So what happens here is that we are not determining the signal's true amplitude. We are computing *a measure of the content* but it is not the actual amplitude. And since we are missing the same exact term form all coefficients, the period, we say that, the Fourier transform determines only relative amplitudes. But often that is good enough. All we are really interested in is the relative levels of powers in the signal. The true power of the signal in most cases where we use these analytical tools is not important. Fourier Spectrum gives us the relative distribution of power among the various harmonic frequencies in the signal. We often normalize the result, putting the maximum at 0 dB. So the relative levels are consistent and useful.

CTFT of aperiodic signals

Now we will take a look at some important **non-periodic** signals and their transforms. We start with the impulse.

Example 4-1

What is the FT of a single impulse function located at origin?

We write the CTFT expression Eq. (1.9) and substitute the delta function for the analysis signal.

$$
\begin{aligned}
&X(\omega)\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} \underbrace{e^{-j(\omega=0)t}}_{=1}\, dt \\
&= 1
\end{aligned}
$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(w)\, e^{-j\omega t}\, dt$$
$$= \int_{-\infty}^{\infty} \delta(w) \cdot e^{-j0t} dt$$
$$= 1$$

In the second step, multipliction by the delta function means to use the value of the function at the origin, and at that point, the value of the complex exonential is 1.0. The integral of the delta function is 1.0 which is the value of the CTFT of a impulse function at the origin. Hence the coefficient is a constant value and we get a flat line for the spectrum.

The other way to think about this is that a delta function is a summation of an infinite number of frequencies, so we see in its decomposition a spectrum that encompass the whole of the frequency space to infinity.

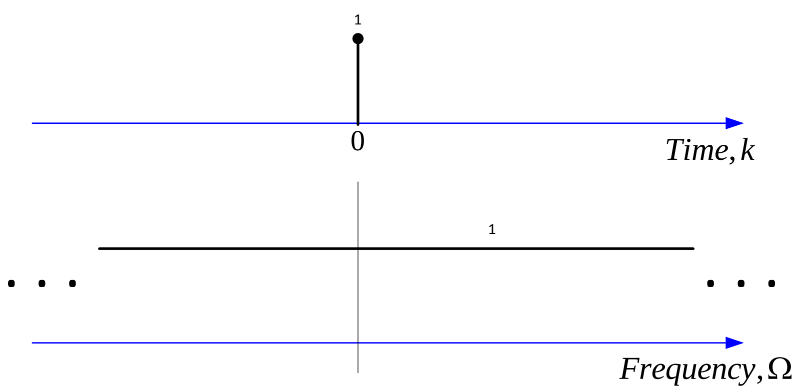


Figure 4.4 – Spectrum of a delta function located at time 0

What happens if there are two impulse functions?

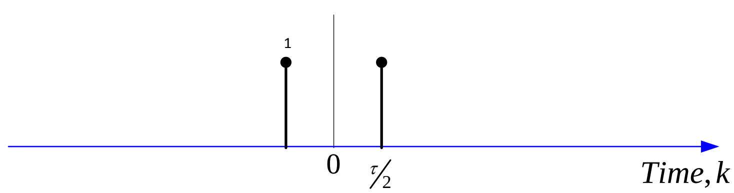


Figure 4.5 – Spectrum of two delta functions

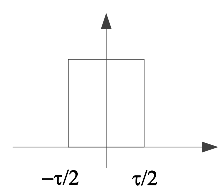
The CTFT calculation can be separated in two parts, one for each of these impulses.

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} \delta(t - \tfrac{\tau}{2}) e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} \delta(t + \tfrac{\tau}{2}) e^{-j\omega t}\, dt \\
&= e^{-j\omega \frac{\tau}{2}} + e^{j\omega \frac{\tau}{2}} \\
&= \frac{1}{2} \quad \cos \quad \omega \tfrac{\tau}{2} \quad - j \sin \quad \omega \tfrac{\tau}{2} \quad + \frac{1}{2} \quad \cos \quad \omega \tfrac{\tau}{2} \quad + j \sin \quad \omega \tfrac{\tau}{2} \\
&= \cos \quad \omega \tfrac{\tau}{2}
\end{aligned}
$$

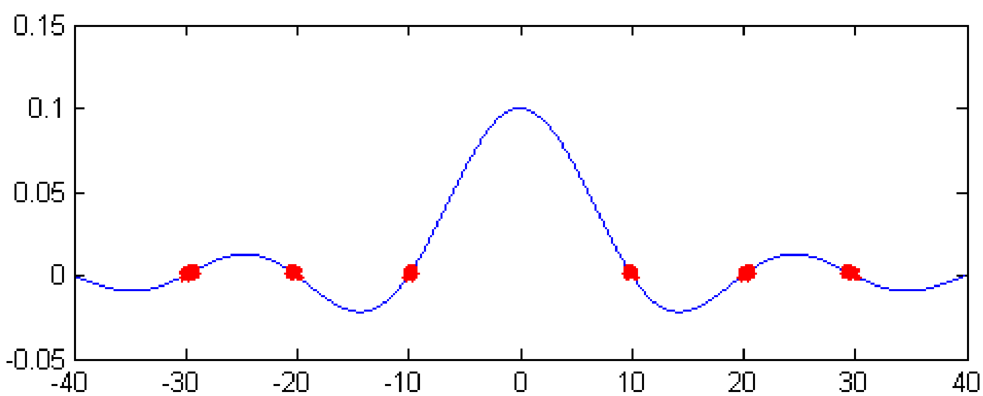
Well, that was kind of obvious. Did we not see in Chapter 2, that two equidistant pulses are the coefficients of a cosine wave. Now can you guess what will happen if we add one more impulse in time domain? As more of these are added, the addition of each new impulse in time-domain makes the overall response a sinc function. We will see this case later.

Example 4-2

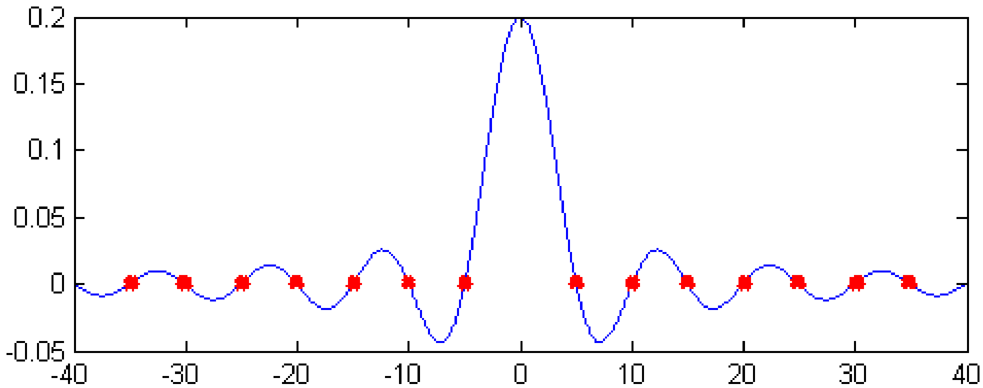
Find the CTFT of a square pulse of amplitude 1v, with a period of τ, located at zero. These are often called either square or rectangular pulses, both names mean the same thing.



(a)



(b)



(c)

Figure 4.6 – Spectrum along a Frequency line
A square pulse has a sinc shaped spectrum. (a) time-domain shape, (b) Spectrum for $\tau = .1$ sec. (c) Spectrum for $\tau = .2$ sec.

```
%Example 4-2
clf
tau = .2;
w = -250: .01: 250;
xom = tau*sinc(w*tau/(2*pi));
plot(w/(2*pi), xom)
```

We write the CTFT as given by Eq. (1.9). The function has a value of 1.0 for the duration of the pulse.

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\ &= \int_{-\tau/2}^{\tau/2} 1 \cdot e^{-j\omega t} dt \\ &= \left. \frac{e^{-j\omega t}}{j\omega} \right|_{-\tau/2}^{\tau/2} \end{aligned}$$

This can be simplified to

$$X(\omega) = \tau \,\text{sinc}\left(\frac{\omega\tau}{2\pi}\right) \tag{1.16}$$

We see the spectrum plotted in Fig-4.4 for $\tau = .1$ and $\tau = .2$ secs. Note that as the pulse gets longer (or wider), its frequency domain spectrum gets narrower. Remember as it is getting narrower, it is approaching the behavior of a delta function.

For first case, the first zero crossing occurs 10 Hz. This is the inverse of the pulse time 0.1. For the second case, when the pulse is .2 seconds wide, the zero crossing occurs at 5 Hz. The spectrum is aperiodic and has infinite harmonics. It is the generalization of the two pulse case in Example 4-1. What is the significance of these zero crossings? Note that the spectrum of the square pulse is given by a sinc function. The sinc function is zero for every integer value of its argument, so we get zeros at these certain frequency points.

If the pulse were to become infinitely wide, the FT would become an impulse function. If it were infinitely narrow as in Example 4-1, the frequency spectrum would be flat. A constant signal in time domain has a delta form in the frequency domain. This bi-directional relationship is often written as:

$$\begin{aligned} 1 &\xrightarrow[CTFT]{} \delta(\omega) \\ \delta(\omega) &\xrightarrow[CTFT]{} 1 \end{aligned} \tag{1.17}$$

Example 4-3

Now assume that instead of the square pulse shown in Example 4-2, we are given a frequency response that looks like a square pulse. The spectrum is flat for a certain band, from $-W$ to $+W$ Hz. Notice, that in the first example, we defined the half width of the pulse as $\tau / 2$ but here we define the half bandwidth by W and not by W/2. The reason is that in time domain, when a pulse is moved, its period is still τ. But bandwidth is designated as a positive quantity. There is no such thing as a negative bandwidth. In this case, the bandwidth of this signal (because it is centered at 0) is said to be W Hz and not 2W Hz. However if this signal were moved to a higher frequency center such that the whole signal was in the positive frequency range, it would be said to have a bandwidth of 2W Hz. This crazy definition gives rise to the concepts of lowpass and bandpass bandwidths.

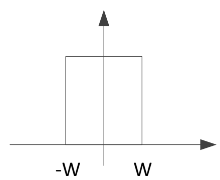
What time-domain signal produces this frequency response? We compute the time domain signal by the inverse Fourier transform.

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-W}^{W} 1 \cdot e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \frac{e^{j\omega t}}{j\omega} \Bigg|_{-W}^{W} \end{aligned}$$

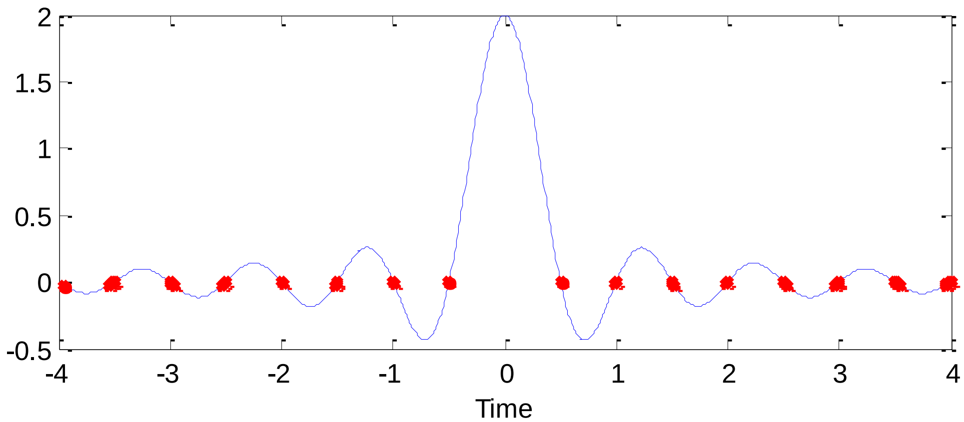
Which can be simplified to

$$x(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{W}{\pi}t\right)$$

Well this also gives us a sinc function. So it looks as if a sinc function in time domain gives a square frequency response. These are shown in Figure 4.7 for two cases of bandwidths. These look like strange shapes for a time domain signal because they are not limited to a certain time period. But because they are "well-behaved", which means they cross zeros at predicable points, we can and do use these as signal shapes to transmit signals. Although theoretically wonderful, the sinc function cannot be used in real systems because it is infinitely long. An alternate raised cosine shape is the most commonly used symbol shape.



(a)



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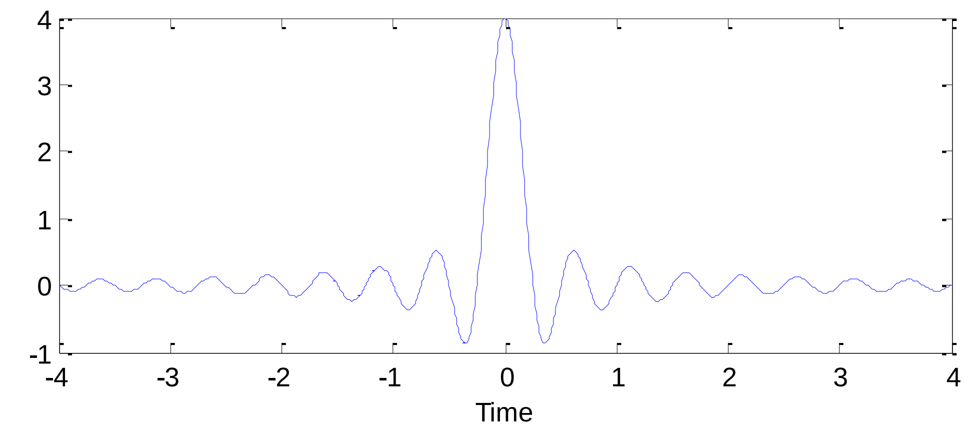


(c)

Figure 4.7 – Time domain signal corresponding to the rectangular frequency. To obtain a rectangular frequency spectrum, a sinc pulse shape is required in time-domain. A narrow band signal is slower than a wideband signal in its zero crossings. (a) W = 2π Hz, (b) W = 4π Hz

The frequency spectrum shown in Fig. 4.7(a) is a very desirable form. We want the frequency response to be tightly constrained. The way to get this type of spectrum is to have a time domain signal that is a sinc function. This is the dual of the first case, where a square pulse produces a sinc frequency response.

For W = 1 Hz, we get the first zero crossing at = 0.5 seconds ($2W = 1/\tau$ and for the second case W = 2 Hz, the first zero crossing occurs at 0.25 seconds. This tells us that a wideband signal requires a faster signal than one that is narrow band. Thes**e** two cases are complementary and very useful.

Example 4-4

Here is an another pair of very important CTFTs.

We have a single impulse located at ω_1 in frequency domain. In Example 4-1, we saw what we get when the impulse is located at zero frequency. Here it is located at a non-zero frequency. What signal gives this FT?

We will take the inverse FT, denoted by this pretty symbol $\Im^{-1}$. We characterize the single impulse as a delta function, $\delta(\omega-\omega_1)$.

$$
\begin{aligned}
x(t) &= \Im^{-1}\left\{\delta(\omega-\omega_1)\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_1)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}e^{j\omega_1 t}\Big|_{-\infty}^{\infty} \\
&= \frac{1}{2\pi}e^{j\omega_1 t}
\end{aligned}
$$

This gives us the complex exponential in time domain. Well, this was kind of obvious too. In Chapter two we looked at the FSC of a complex exponential. Because it is a complex signal, it has non-symmetrical frequency response which consists of just one impulse located at the exponential frequency. Fourier transform gives the same result.

We can write the result by taking the 2π factor to the other side.

$$2\pi\,\delta(w-\omega_1)\xrightarrow[CTFT]{} e^{-j\omega_1 t} \quad \textit{Frequency to Time}$$
$$2\pi\,\delta(t-T_1)\xrightarrow[CTFT]{} e^{-j\frac{2\pi}{T_1}} \quad \textit{Time to Frequency} \tag{1.18}$$

These results are very important and should be committed to memory.

Example 4-5

What is the FT of a cosine wave?

We are doing FT, so we make cosine wave a non-periodic signal by limiting it to one period.

$$\begin{aligned}
x(t) &= \Im^{-1}\{\cos\omega_0 t\} \\
&= \frac{1}{2\pi}\int_0^{-2\pi}\frac{1}{2}\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_0^{-2\pi}\frac{e^{j(\omega+\omega_0)t}}{2}d\omega+\frac{1}{2\pi}\int_0^{-2\pi}\frac{e^{j(\omega-\omega_0)t}}{2}d\omega
\end{aligned}$$

From example 4-4, we get this transform.

$$2\pi\delta(\omega+\omega_0)\Leftrightarrow\frac{1}{2\pi}\int_0^{-2\pi}e^{j(\omega+\omega_0)t}\,d\omega$$

Substituting this transform for the exponentials, we get

$$\begin{aligned}
&= \frac{1}{2\pi}\int_0^{-2\pi}\frac{e^{j(\omega+\omega_0)t}}{2}d\omega+\frac{1}{2\pi}\int_0^{-2\pi}\frac{e^{j(\omega-\omega_0)t}}{2}d\omega \\
&= \pi\delta(\omega+\omega_0)+\pi\delta(\omega-\omega_0)
\end{aligned}$$

The only difference we see between the FT of cosine wave and FSC we computed in Chapter 2 seems to be the scaling. In the case of FSC, we got two delta functions at $-\omega_0$ and $+\omega_0$ of amplitude ½. The amplitude of the FT computed is π. So we seem to be off by 2π when comparing the FSC of a signal with its FT. We explain the reason for this later in this chapter.

Fourier transform of a periodic signal

Fourier transform came about so that the analysis can be made rigorously applicable to non-periodic signals. All of the above signals were aperiodic, even Example 4-5, where we did the CTFT of a cosine wave but we limited it to one period.

Can we also use the FT for a periodic signal? That would sure simplify things. We can then go ahead and forget about Fourier series. But will we get the same answer as with the Fourier series?

Let's take a periodic signal x(t) with fundamental frequency of $\omega_0 = 2\pi / T_0$ and write its FS representation.

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t} \tag{1.19}$$

Where C_n are the CTFS coefficients and are given by

$$C_n = \frac{1}{T_0} \int_{T_0} x(t) \, e^{-jn\omega_0 t} dt \tag{1.20}$$

Let's take the CTFT of both sides of Eq. (1.19) using the symbol $\Im$ to indicate a FT, we get

$$X(\omega) = \Im\{x(t)\} = \Im\left\{ \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t} \right\} \tag{1.21}$$

We can move the coefficients out because they are not a function of frequency. They are just numbers.

$$X(\omega) = \left(\sum_{n=-\infty}^{\infty} C_n \right) \Im\left\{ e^{jn\omega_0 t} \right\} \tag{1.22}$$

The FT of $e^{jn\omega_0 t}$ is a delta function as we learned in Example 4-4. Making the substitution, we get

$$X(\omega) = 2\pi \left(\sum_{k=-\infty}^{\infty} C_k \right) \delta(\omega - n\omega_0) \tag{1.23}$$

What does this equation say? It says that the ***CTFT of a periodic signal*** is a sampled version of the Fourier series coefficient of the periodic case. The coefficients from the Fourier series of the same periodic series are multiplied by a train of impulses. This results in again a discrete spectrum with the area of each impulse at the $n\omega_0$ harmonic frequency equal to the Fourier series coefficient of that frequency times 2π.

So now we have the **Fourier Transform of a periodic signal** as a discrete form of the Fourier series coefficients. Okay, this is admittedly strange. The FT of a non-periodic signal is continuous but the FT of a periodic signal is discrete? Yes. That is how it is. The delta function in (1.23) combs/sifts the coefficients and then repeats them with fundamental frequency ω_0.

CTFT of an aperiodic signal → aperiodic and continuous
CTFT of a periodic signal → discrete and periodic.

Remember, we said that the CTFS does not exactly measure the true "quantity" of each harmonic and in Eq. (1.23) we see the proof. The CTFT values are actually 2π times greater and this explains the results of Example 4-5.

CTFT of <u>periodic</u> signals

Now we will look at some periodic signals and their Fourier transform. The FT of periodic signals is given as a modification of the Fourier series coefficients by

$$X(\omega) = 2\pi\left(\sum_{k=-\infty}^{\infty} C_k\right)\delta(\omega - n\omega_0)$$

Example 4-6

What is the FT of a ***periodic*** impulse train with period T_0.

We already computed the FSC of an impulse. It is a constant. The FT of an impulse train can be obtained from the relationship derived between the FSC and FT, Eq. (1.23).

$$X(\omega) = 2\pi\left(\sum_{k=-\infty}^{\infty} C_k\right)\delta(\omega - k\omega_0)$$

$$= 2\pi\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_0)$$

The FSC of a single delta function is 1, a flat line. The result above samples (also called sifting because it is kind of like passing it through a sieve with a lot of holes in it.) that flat line in frequency domain, resulting in an impulse train, with each impulse repeating at the fundamental frequency of the signal, $F_0 = 1/T_0$.

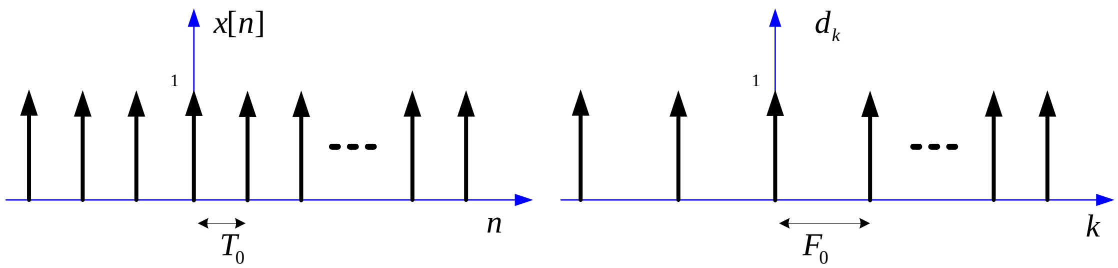


Figure 4.8 – An impulse train and its discrete-time Fourier coefficients

Example 4-7

Find the Fourier transform of a ***periodic*** square pulse train.

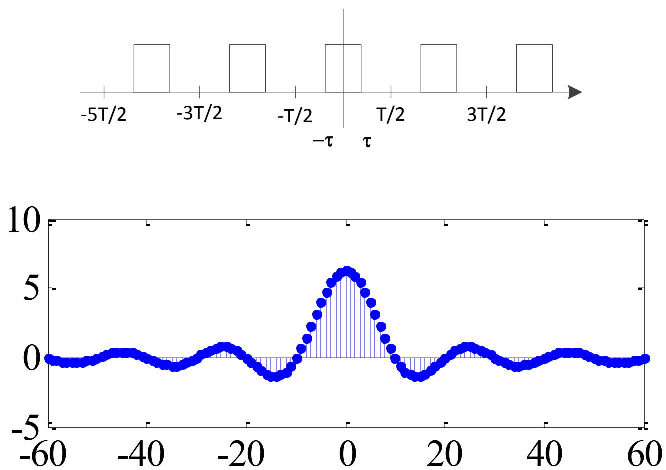


Figure 4.9 – A square pulse train and its discrete-time Fourier coefficients

The FSC of a square pulse train is given by (See Chapter 2)

$$C_k = \frac{\tau}{T}\text{sinc}\left(\frac{k\tau}{T}\right)$$

For the FT of this periodic signal, we will use Eq. (1.23)

$$X(\omega) = 2\pi\left(\sum_{k=-\infty}^{\infty} C_k\right)\delta(\omega - k\omega_0)$$

The result is essentially the sampled version of the Fourier series coefficients scaled by 2π (see last example, Chapter 2) which are of course themselves discrete. So here we multiply a set of discrete numbers by an impulse train to obtain a sampled version of the coefficients. These are shown in Fig. 4.9

Discrete-time Fourier transform (DTFT)

The concept of Fourier transform we discussed for a continuous-time aperiodic signal applies equally to discrete-time aperiodic signals except for some changes in the integration period. The discrete-time Fourier transform, also called the DTFT synthesis equation is given by

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[k] e^{-j\Omega nk} \tag{1.24}$$

Here n is the index of the harmonics and k the index of time. (Note that many books use the reverse notation, with n for time and k for harmonic index. So please make note that your homework/book may have different notation.) Recall that for the continuous case we refer to the fundamental frequency as $\omega_0 = \frac{2\pi}{T_0}$, and for the discrete case as $\Omega_0 = \frac{2\pi}{K_0}$. The terms ω and Ω are the continuous vrsion of the analog and the discrete frequencies.

The inverse discrete-time Fourier transform, also called the analysis equation is given by

$$x[k] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} \, d\Omega \tag{1.25}$$

Note that index n goes with th frequency. These two equations (1.25) and (1.24) are called the discrete-time Fourier transform pair and are also written as:

$$x[k] \Leftrightarrow X(\Omega) \tag{1.26}$$

Both the continuous-time and the discrete-time Fourier series coefficients are discrete. The coefficients of the discrete-time Fourier transform (DTFT) however are continuous, just as they are for the continuous-time case.

We learned in Chapter 3 that discrete signals produce a spectrum that repeats with fundamental frequency. The Fourier Transform of the discrete signals also repeats with the fundamental frequency, the only difference being that the DTFT is continuous where the DTFS spectrum is discrete.

CTFT of an aperiodic signal → **aperiodic and continuous**
CTFT of a periodic signal → **discrete and periodic.**
DTFT of an aperiodic signal → **periodic and continuous**
DTFT of a periodic signal → **discrete and periodic.**

DTFT of aperiodic signals

Example 4-8

Find the DTFT of this signal.

So here we now look at three impulses as opposed to just two in Example 4-1. In that case, the result was a cosine function.

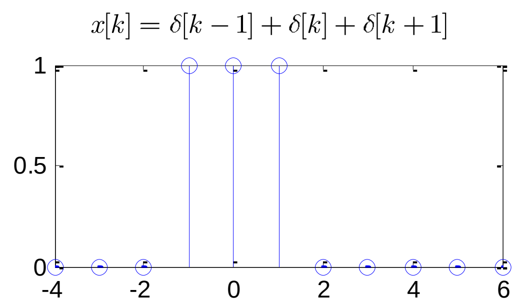


Figure 4.10 – Signal of example 4-8

We compute the DTFT by treating each impulse individually.

$$X(\Omega) = \sum_{k=-\infty}^{\infty} \delta(k-1)e^{-j\Omega k} + \sum_{k=-\infty}^{\infty} \delta(k)e^{-j\Omega k} + \sum_{k=-\infty}^{\infty} \delta(k+1)e^{-j\Omega k}$$

$$= 1 +^{-j\omega} + e^{j\omega}$$

$$= 1 + 2\cos\omega$$

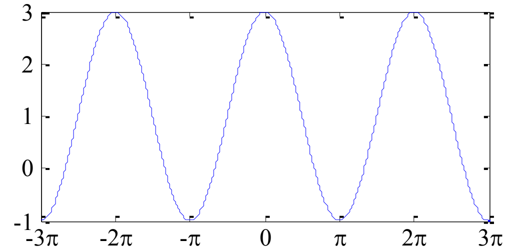


Figure 4.11 – The DTFT of signal 4-8 (a) DTFT

Note that the spectrum is periodic with period 2π.

Here we also get a cosine wave. What happens if we have four impulses? Would we still get a cosine?

Example 4-9

What is the DTFT of this discrete signal?

We can again treat each one of these individually.

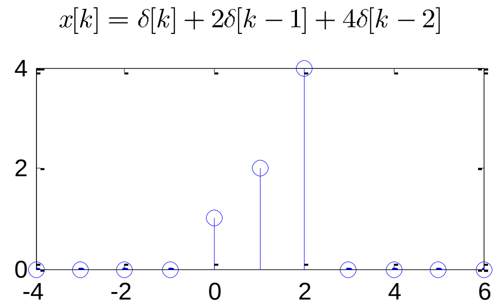
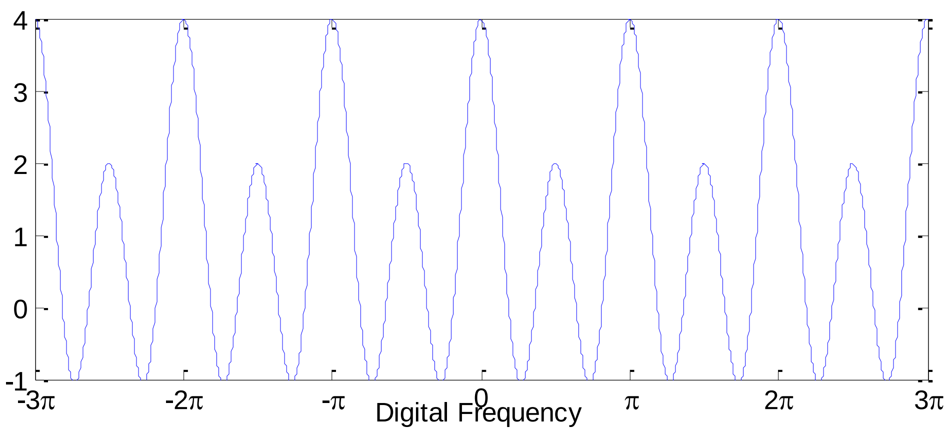


Figure 4.12 – Signal of example 4-9

$$X(\Omega) = \sum_{k=-\infty}^{\infty} \delta(k) e^{-j\Omega k} + \sum_{k=-\infty}^{\infty} 2\delta(k-1) e^{-j\Omega k} + \sum_{k=-\infty}^{\infty} 4\delta(k-2) e^{-j\Omega k}$$

$$\begin{aligned} &= 1 + 2e^{-j2\omega} + 4e^{j4\omega} \\ &= 1 + \cos(2\omega) - j\sin(2\omega) + 2\cos(4\omega) - 2j\sin(4\omega) \\ &= \underbrace{1 + \cos(2\omega) + 2\cos(4\omega)}_{\mathrm{Re}\,al} - j\underbrace{(\sin(2\omega) + 2\sin(4\omega))}_{\mathrm{Im}\,ag} \end{aligned}$$



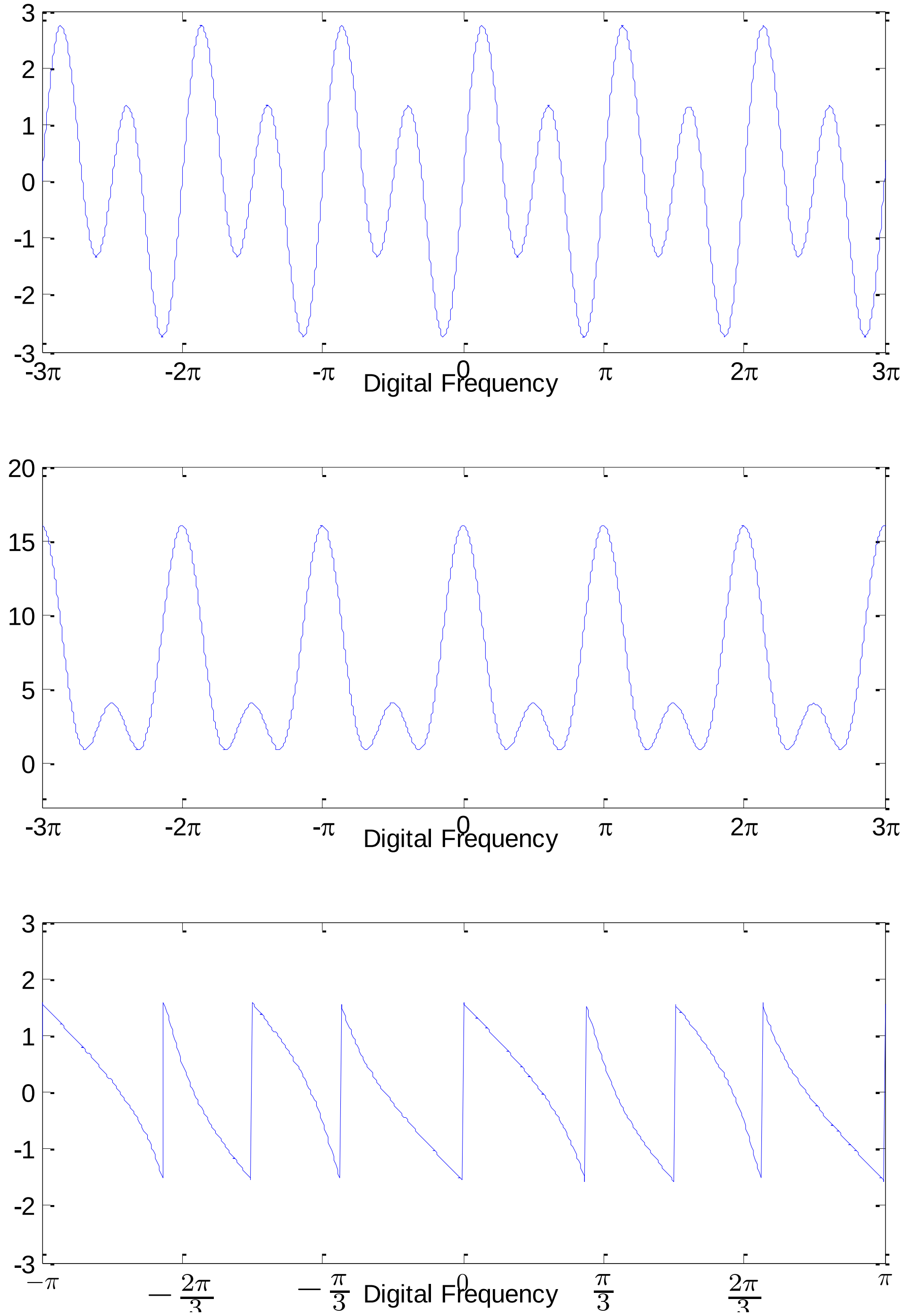


Figure 4.13 – (a) Real portion of the DTFT (b) Imaginary part of the DTFT, (c) the power spectrum of the signal, (d) the phase spectrum of the signal.

We also have three pulses here although of unequal magnitude. But the spectrum is still a form of a cosine wave. A spectrum is usually plotted as the square of magnitude of the signal. Here we plot the power spectrum and the phase spectrum of the signal.

I have been giving a short shrift to the phase. The reason is that in a majority of the cases, phase is not very instructive. In practical sense, there is not much we can do to control phase. For communication signals, it is the amplitude or the power spectrum that gives the information that we want and need. However for radar applications, phase information is very important and can be used to determine both the motion and Doppler shift of the signal. So phase is not always useless.

Example 4-10

Find the DTFT of this signal.

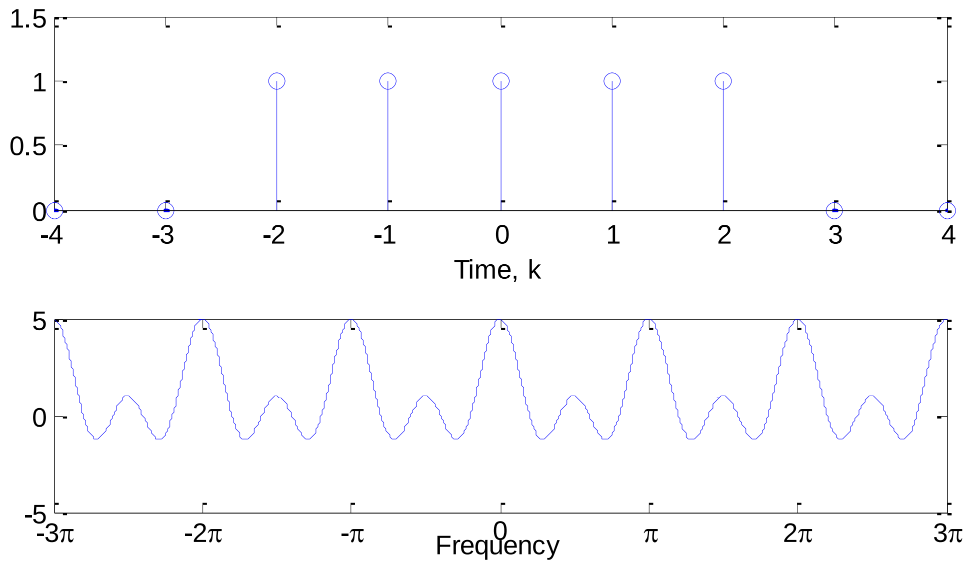


Figure 4.14 – A pulse of length N = 5 and its spectrum

```
% Example 4-10
clf
Kp = 5;
a = [1];
b = [ 0 0 a a a a a 0 0 ];
subplot(2,1,1)
n = -4: 4;
stem(n, b)
xlabel('Time, k')
axis([-4 4 0 1.5])
subplot(2,1,2)
t = -10:.01: 10;
xom = (Kp)*diric(2*t, Kp);
grid on
```

```
plot(t, xom)
pilabels
xlabel('Frequency')
hold on
```

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{k=-N}^{N} 1e^{-j\Omega n}$$

$$= \frac{\sin\left(\frac{2N+1}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}$$

Here we have five impulses. The spectrum equation looks like a sinc function but it is instead a variation, called the Drichlet function. The Drichlet function is essentially a repeating or periodic sinc function. With size of pulse greater than 3, we begin to see a repeating sinc function as the spectrum of such signals.

The DTFT just as is the CTFT is continuous. But it has an additional property that we mentioned in relation to the discrete Fourier Series. The spectrum obtained by a DTFT also replicates at the fundamental frequency of the discrete signal. In Fig. 4-14, the spectrum has a period of 2π.

DTFT of <u>periodic</u> signals

Just as we were able to derive a method for applying the CTFT to periodic signals, we can do the same with discrete-time Fourier transform. Let's take a periodic signal with period K_0 and write its discrete Fourier series equation.

$$x[k] = \sum_{K_0} C_n \, e^{jn\Omega_0 k}$$

Where the coefficients are given by

$$C_n = \frac{1}{K_0} \sum_{K_0} x[k] e^{-jn\Omega_0 k}$$

FT of this periodic signal is also given by the same relationship we derived for continuous signals in Eq. (1.24). The DTFT of a periodic signal is also the sampled version of the discrete-time Fourier series coefficients, just as for the continuous time case.

$$X(\Omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n \, \delta\left(\Omega - \frac{2\pi n}{K_0}\right) \qquad (1.27)$$

Here the coefficients (which come from the DTFSC) repeat with the fundamental frequency $\Omega_0 = 2\pi / K_0$.

The DTFT of aperiodic signals is continuous. In most cases, the DTFT is obtained by closed from analysis, and not by computers. This is because the computer solutions are by necessity, discrete. So we have two issues we want to address, how to compute the DTFT of a signal numerically and two how to make the DTFT apply to periodic signal.

We showed that the DTFT of periodic signals are continuous and repeating. The same signal if made periodic has the same coefficients except they are discrete per Eq. (1.27). The reason is somewhat intuitive. The periodicity means that the signal information is repeating with some frequency. As you recall we said that for discrete signals, the harmonics of the digital frequency are identical. They do not give us any unique information. All the information comes from harmonics that are instead inside the fundamental period.

For the case of periodic discrete signals, we take the same coefficients from the aperiodic case, but now add an additional condition of a repeating period. But did we not say that the coefficients of the aperiodic signal repeat? So what is different here? What is the relationship of the frequency at which the aperiodic coefficients repeat vs. the periodic signal. If the aperiodic signal has an infinite period, then what is that frequency at which the DTFT was repeating?

Now here is where we get tricky. The period of the DTFT for the aperiodic signals is 2π. This is kind of a generic period. Not related to any frequency per say. So now if the signal is actually repeating let's say at a frequency of Ω_0, then the DTFT will repeat instead of at 2π radians, but at $\Omega_0 \pm 2\pi$. So that is the difference between the aperiodic and periodic signal repeat frequencies. One repeats with 2π and the other with addition of the signal digital frequency.

The periodic-ness of the signal results in a form of sampling of the DTFT. The only difference one sees between the DTFT of an aperiodic vs. a periodic signal is that the DTFT of the periodic signal is discrete, being a sampled version of the discrete-time Fourier series coefficients.

Example 4-12

Find the FT of the periodic impulse train.

$$x[k] = \sum_{n=-\infty}^{\infty} \delta(n - Nk)$$

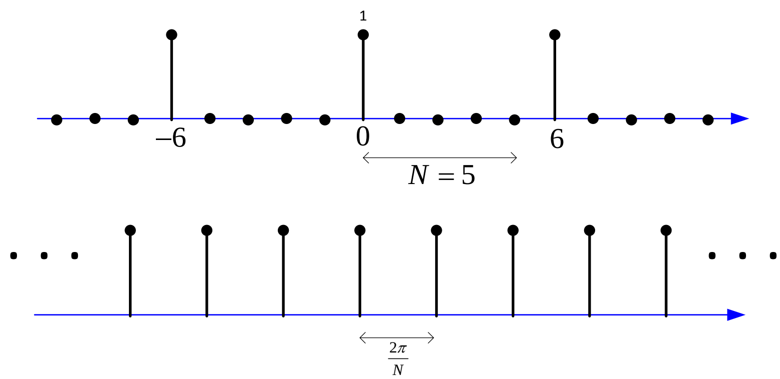


Figure 4.15 – A pulse train and its spectrum

The Fourier coefficients of this signal are given by

$$d_k = \sum_{k=K_0} x[k] e^{-j\, n\Omega_0\, k} = \frac{1}{K_0}$$

The FT is given by

$$X(\Omega) = \frac{2\pi}{K_0} \sum_{n=-\infty}^{\infty} \delta\ \Omega - n\Omega_0$$

The FT is plotted for N = 5. The transform repeats at the fundamental frequency which is $\frac{2\pi}{5}$.

Example 4-12

Find the DTFT of $x[k] = \cos(\Omega_0 k)$.

We write this signal in its Euler form as

$$\cos(\Omega_0 k) = \frac{1}{2} e^{j\Omega_0 k} + \frac{1}{2} e^{-j\Omega_0 k}$$

Assume that $\Omega_0 = \frac{2\pi}{5}$

The coefficients of the signal are $\frac{1}{2}$ at k = ± 1. Applying Eq. (1.27) to the coefficients, we get for $-\pi \le \Omega < \pi$

$$X(\Omega) = \pi\delta\left(\Omega - \frac{2\pi}{5}\right) + \pi\delta\left(\Omega + \frac{2\pi}{5}\right)$$

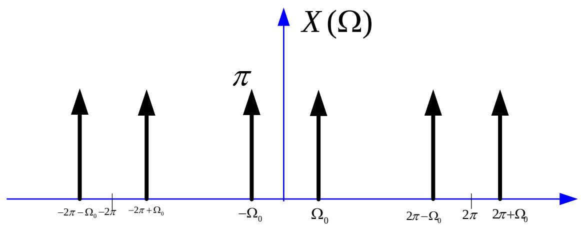


Figure 4.16 DTFT of a discrete cosine wave

The result is that the coefficients are repeating with frequency Ω_0. This is identical in form to the DTFSC.

Example 4-13

Find the DTFT of this discrete periodic signal.

$$x[k] = e^{j\Omega_0 k}$$

This is the complex exponential of a specific frequency. We will use Eq. (1.27) to find the DTFT of this signal. The DTFSC of this signal we know from Example 3-9, Chapter 3. The coefficients already repeat with frequency Ω_0.

$$D_n = \begin{cases} 1 & n = p\Omega_0 \\ 0 & elsewhere \end{cases}$$

We write the DTFT of this signal as

$$X(\Omega) = 2\pi \sum_{n=-\infty}^{\infty} \delta \;\; \Omega - \;\; \Omega_0 + 2\pi m$$

The coefficients were repeating to start with, with frequency Ω_0. So this operation did nothing new . The DTFT is same as the DTFS coefficients.

Example 4-13

Find the DTFT of this discrete periodic signal. The signal is periodic with period K_0. The length of the impulses is K_p samples.

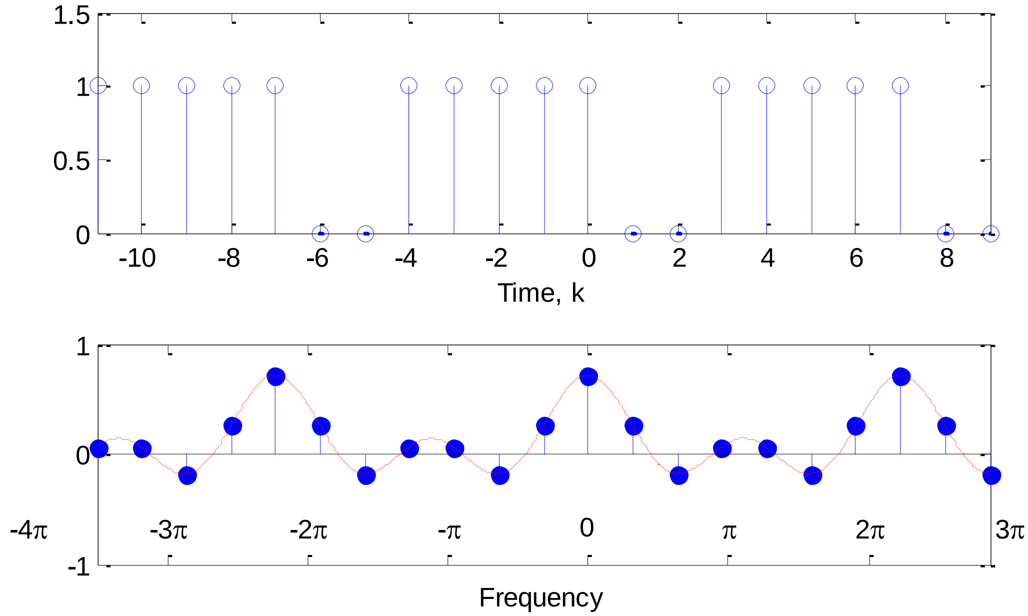


Figure 4-17 DTFT of a periodic signal (a) time domain signal, (b) DTFSC of the signal.

The DTFSC of this signal are discrete and are given by

$$\text{Real } C_n = \frac{1}{K_0}\left[\frac{\sin\ K_p n\pi / K_0}{\sin\ n\pi / K_0}\right]$$

The coefficients are shown in Figure 4.17 for Kp = 5 and K0 = 7 .

To obtain the DTFT of this signal means multiplying the DTFSC with a pulse train of frequency $\Omega_0 = 2\pi / 7$. But these coefficients are already located at a frequency resolution of $\Omega_0 = 2\pi / 7$. How do we know that? Just count the number of samples from $-\pi$ to π. We get 7. So the fundamental digital frequency is $2\pi / 7$.

So the DTFT of this periodic signal is same as DTFSC except it is scaled by a factor 2π.

Let's revisit all the cases that we looked at so far.

Continuous-time Fourier Series

We started with this case in Chapter one. The CTFS is defined for periodic signals where time is continuous, such as the signal shown below.

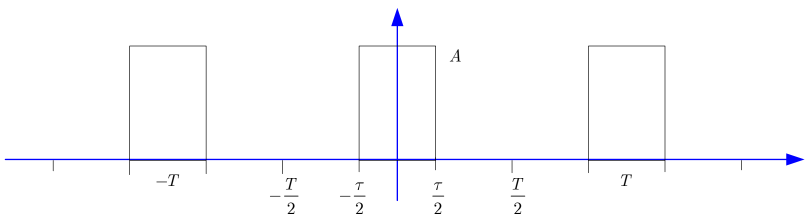


Figure 4.18 A periodic signal with continuous time

The CTFS coefficients of this periodic signal are given by

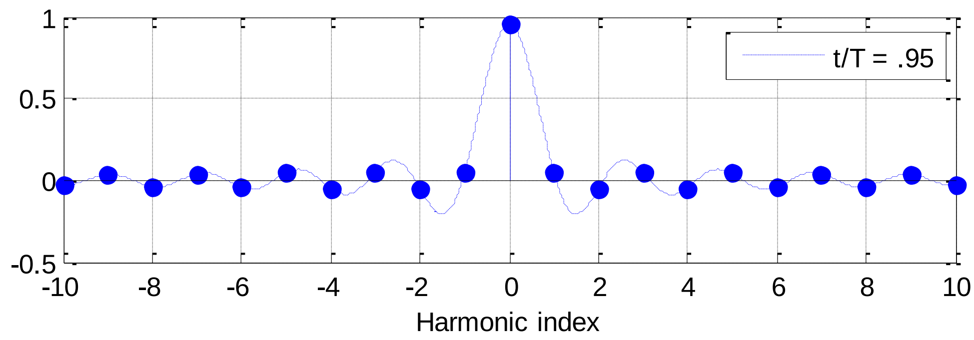
$$C_n = \frac{\tau}{T}\text{sinc}\left(\frac{n\tau}{T}\right)$$

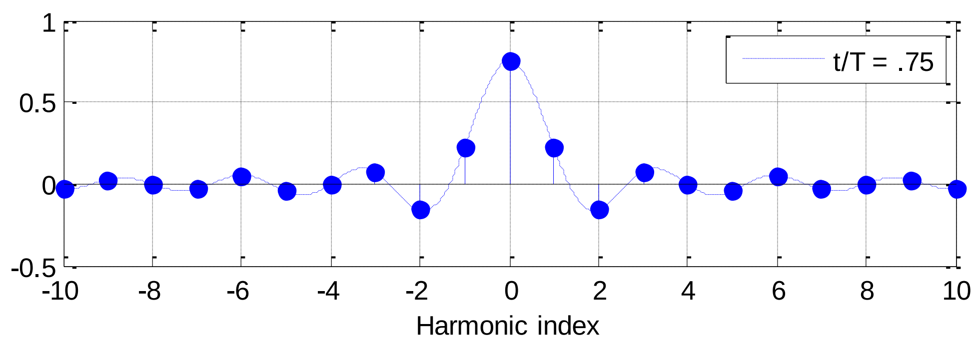
Assume T = $T = 1/2\pi$ and for case 1 $\tau = T/2 = 1/4\pi$. Let's examine this spectrum. First thing we notice is that the spectrum is a sinc function which we know does not repeat. The fundamental frequency of this signal is $\omega_0 = 1/T = 2\pi$. For case 1, when $\tau/T = .5$, The value of the spectrum at f = 0, has a value of .50, which is the value of the dc component or is equal to $\frac{\tau}{T}$.

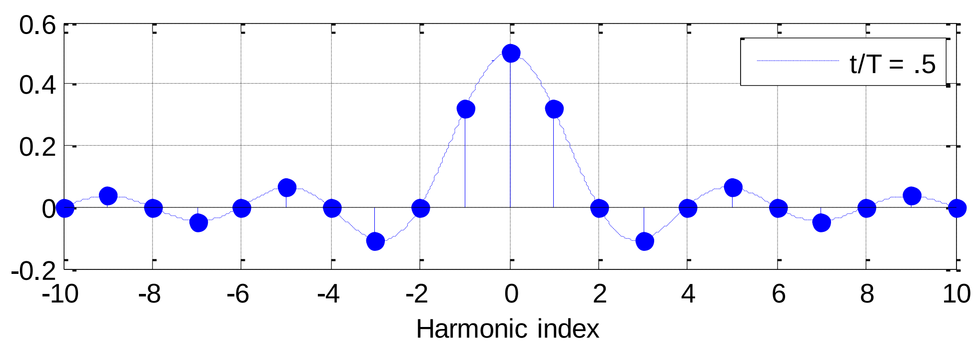
At n = 2, the spectrum shows a zero value which is what you get for sinc(1). The sinc function is zero at all integer values of its argument. This corresponds to a frequency of 2π. The spectrum hence is zero at n = $\pm 2, \pm 4, \pm 6, \cdots$ which corresponds to frequencies of $\pm 2\pi, \pm 4\pi, \pm 6\pi, \cdots$.

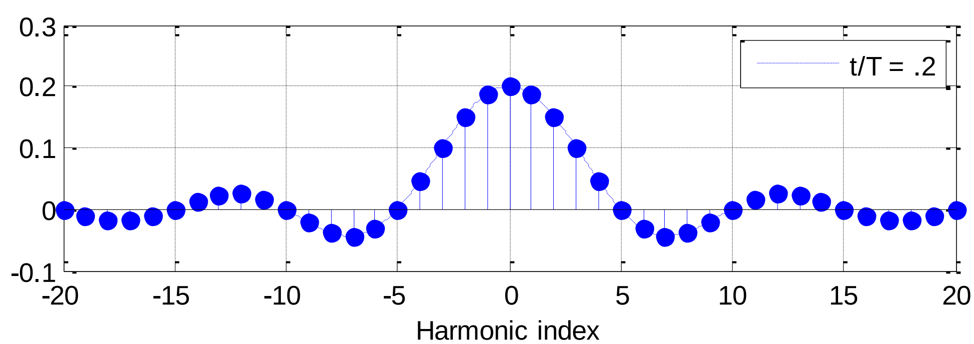
For case $\tau/T = .2$ where the argument of the sinc function is $n/5$, the spectrum is zero at $n = \pm 5, \pm 10, \pm 15, \cdots$ which corresponds to frequencies of $\pm 5\pi, \pm 10\pi, \pm 15\pi, \cdots$.

Note that as the $\tau/T \to 1$, the spectrum begins to look like an impulse function, which is exactly the spectrum of an impulse train.


t/T = .95
1
0.5
0
-0.5
-10 -8 -6 -4 -2 0 2 4 6 8 10
Harmonic index


t/T = .75
1
0.5
0
-0.5
-10 -8 -6 -4 -2 0 2 4 6 8 10
Harmonic index


t/T = .5
0.6
0.4
0.2
0
-0.2
-10 -8 -6 -4 -2 0 2 4 6 8 10
Harmonic index


t/T = .2
0.3
0.2
0.1
0
-0.1
-20 -15 -10 -5 0 5 10 15 20
Harmonic index

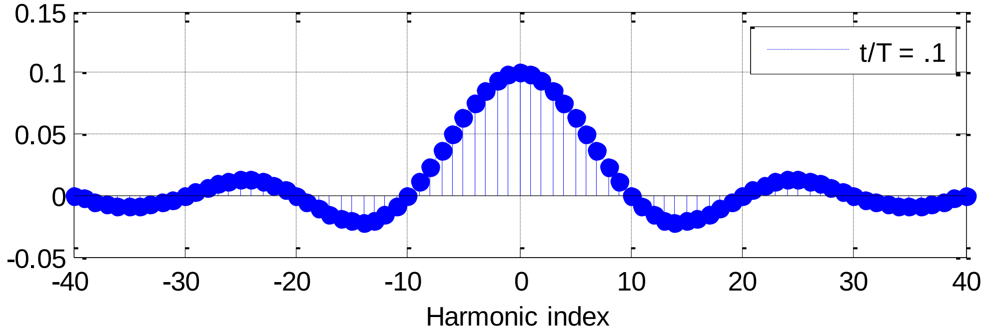


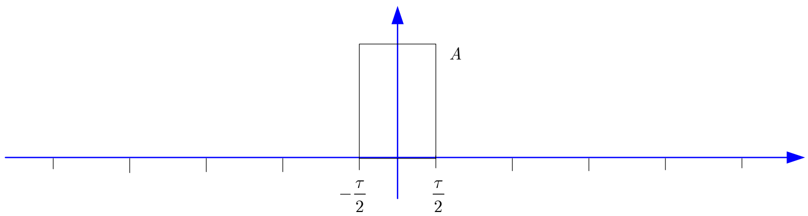
Figure 4.19 The discrete coefficients of the continuous signal as a function of the duty cycle of the signal. As the pulse gets narrow, its CTFSC get more dense.

```
% Figure 4-19
clf
tT = .1;
n = -10: .01: 10;
xom = tT*sinc(n*tT);
plot(n, xom, '-.')
grid on
hold on
k = -40: 40;
xomd = tT*sinc(k*tT);
stem(k, xomd, 'filled')
xlabel('Harmonic index')
legend('t/T = .1')
```

The ratio τ / T is called the duty cycle of the signal. A small duty cycle means that energy is concentrated in much smaller period of time which means that more frequencies are required to represent it (it is approaching a delta function!) A duty cycle of .5 means that the energy is less concentrated. This case has the narrowest main lobe of all cases shown, i.e. this case requires the least amount of bandwidth. As duty cycle decreases, the spectrum gets wider.

Continuous-time Fourier Transform

No let’s look at just one period of the same signal. It is aperiodic, so we do a Fourier transform of this signal.



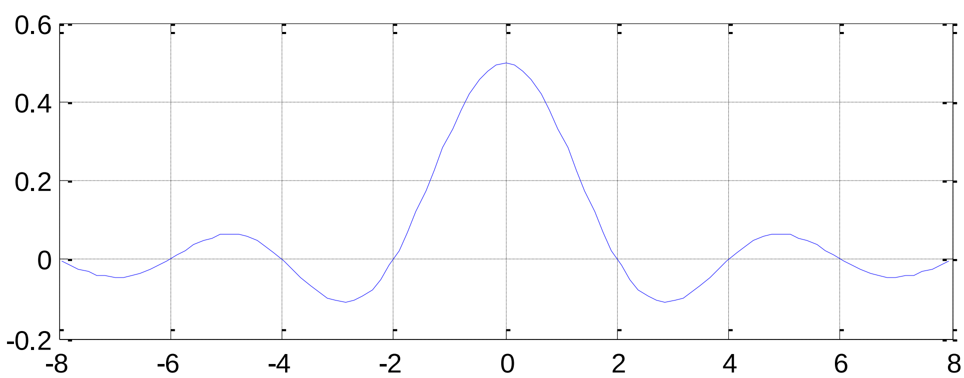
The CTFT of this signal is given by the continuous function

$$X(\omega) = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

Assume that $\tau = \pi$. We plot the CTFT below. Compare this to the expression for the periodic case above. This signal is continuous but also aperiodic. The zeros occur every 2 Hz, why? For $\tau = \pi$, we get

$$X(\omega) = \pi\,\mathrm{sinc}\left(\frac{\omega}{2}\right)$$

So for this case, the sinc function, hence the spectrum is zero for all interer multiples of 2 Hz. Similalry for $\tau = \frac{2\pi}{5}$, we get crossings every 5 Hz. And for $\tau = \frac{\pi}{5}$, we get zero crossing every 10 Hz.



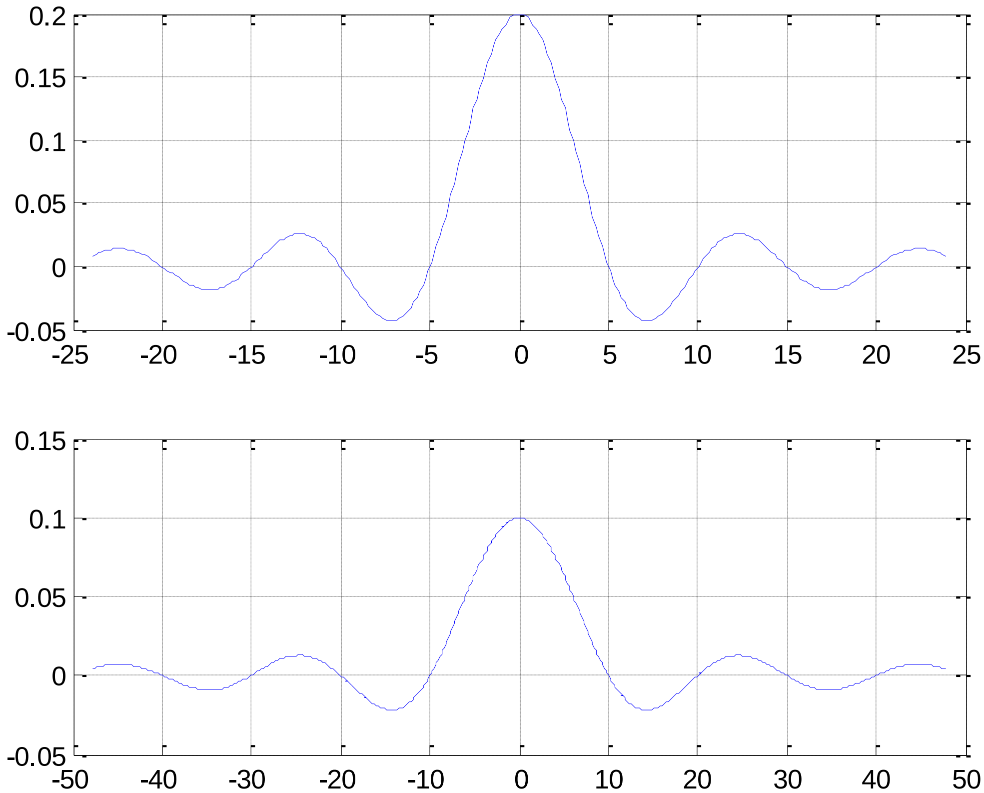


Figure 4.20 The CTFT of the continuous but aperiodic square pulse as a function of the width the square pulse. As the pulse gets narrow, its lobes in the spectrum get wider.

```
% Figure 4.20
clf
tT = .50;
n = -50: 1: 50;
xom = tT*sinc(n*tT/(2*pi));
plot(n/(2*pi), xom)
grid on
```

Here the spectrum is plotted as function of the frequency. As the time duration of the pulse narrows, the signal content spreads. The shape is the same as that of the CTFS case. Both are a non-repeating sinc function, with zero crossings at integer multiple of the fundamental frequency of the signal, main difference being that the first is discrete and the second continuous.

Discrete-time Fourier series, DTFS

Now let's look at the same signal, in a discrete form. We will use the discrete version of the signal shown in Figure 4.18.

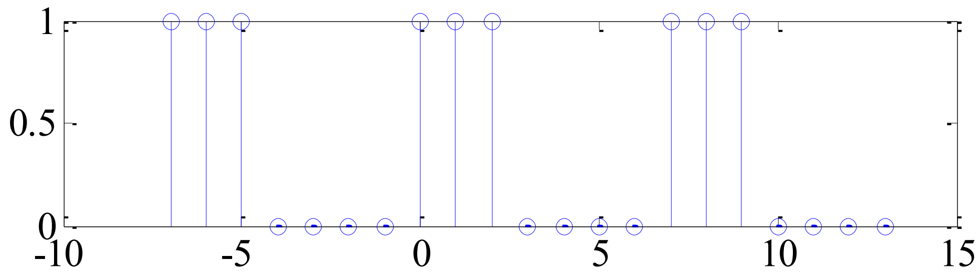


Figure 4.21 The discrete-time periodic signal.

The DTFS coefficients of this signal are computed in Chapter 3, Example 3-8. The spectrum contains a Drichlet function.

Here we plot the spectrums for several cases of pulse sizes while keeping the period fixed. We want to see what effect this has on the spectrum.

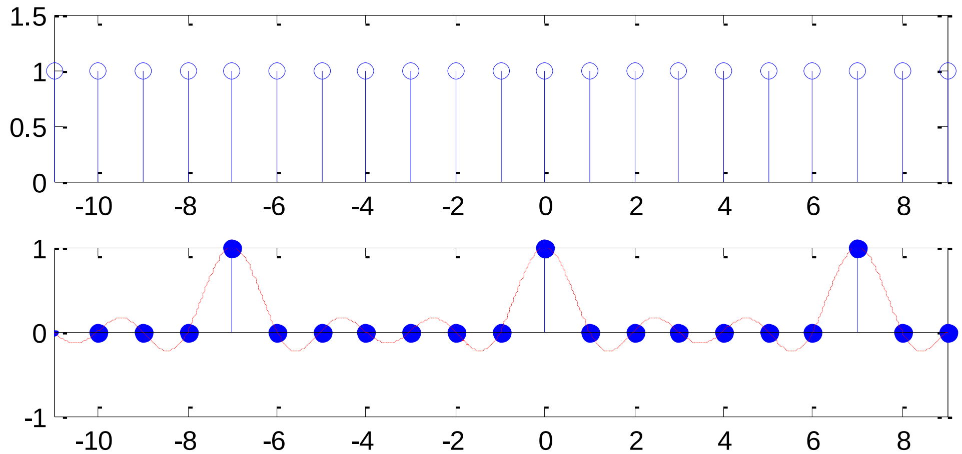


Figure 4.22 The discrete-time periodic signal and its DTFSC - pulse size $K_p = 7, K_0 = 7$

.

In this case, we essentially have an impulse train. The period is 1/7 seconds and as such in the frequency domain, we get impulses located 7 bins apart, each of which are $2\pi / 7$ Hz apart. Hence in the frequency domain the frequency pulses have a period of 2π which corresponds to the period of 7 samples.

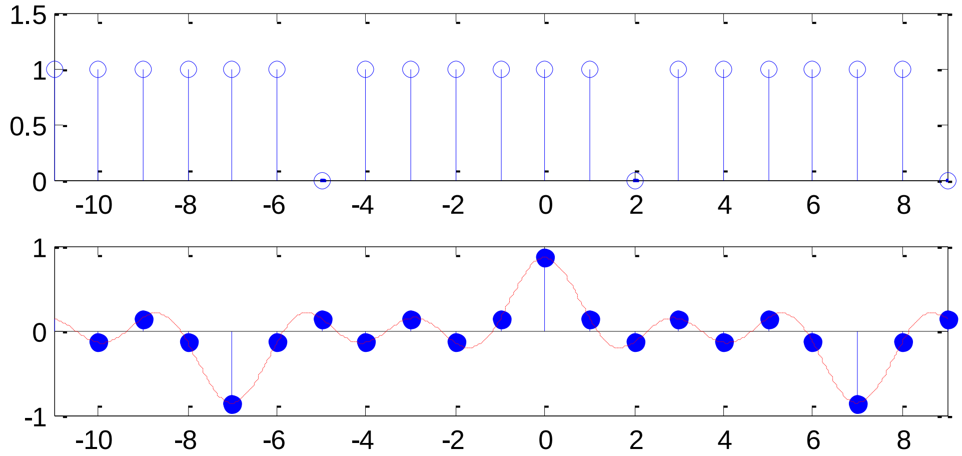


Figure 4.23 The discrete-time periodic signal and its DTFSC - pulse size $K_p = 6, K_0 = 7$

In this case, the pulse size is 6 samples in time domain lasting 6/7 sconds. In frequency domain each bin is $2\pi / 7 = .897 Hz$. We see in the frequency domain that the main lobe is a little over 1 bin wide, which is a bandwith of 7/6 Hz, and is equal to the inverse of the pulse duration time. A wide pulse has narrow bandwidth so in the available frequency space of 2π, we see a sinc like pattern such as we would expect having seen the results from the discrete case. As long as the pulse width is $> \pi$, we will see the sinc function tails.

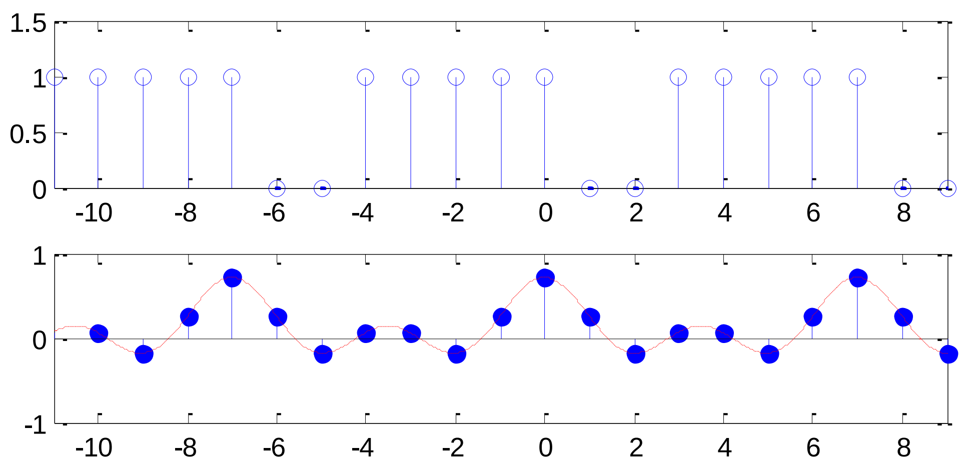


Figure 4.24 The discrete-time periodic signal and its DTFSC - pulse size $K_p = 5, K_0 = 7$

In this case, the pulse size is 5 samples in time domain lasting 5/7 sconds. In the frequency domain the main lobe is approximately 1.5 bins wide, from which we get a bandwidth of 1.35 Hz, which is pretty close to actual the bandwith of 7/5 Hz = 1.4 Hz, the inverse of the pulse duration time. We still see a sinc like pattern since the width of the pulse is still $> \pi$.

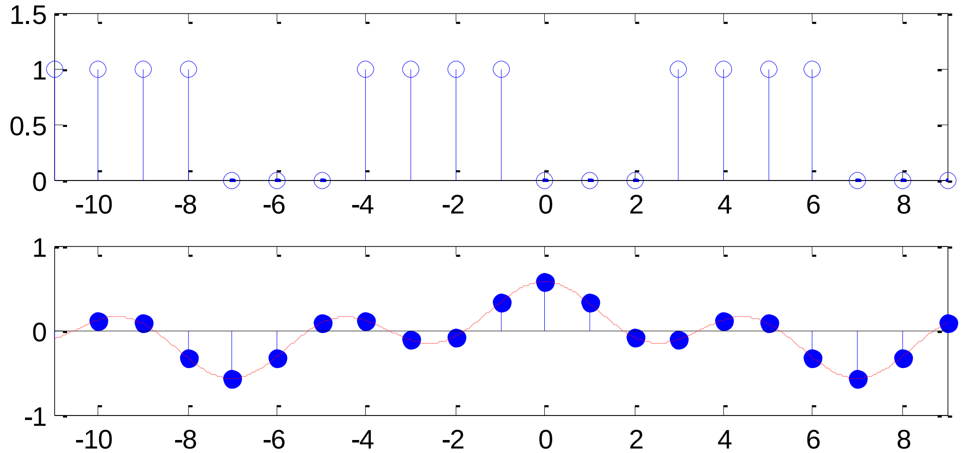


Figure 4.25 The discrete-time periodic signal and its DTFSC - pulse size $K_p = 5, K_0 = 7$

In this case, the pulse size is 4 samples in time domain lasting 4/7 sconds. In the frequency domain that the main lobe is approximately 2 bins wide, from which we get a bandwidth of 1.79 Hz, which is pretty close to actual the bandwith of 7/4 Hz = 1.75 Hz, the inverse of the pulse duration time. The sinc like tails are enow beginning to disappear.

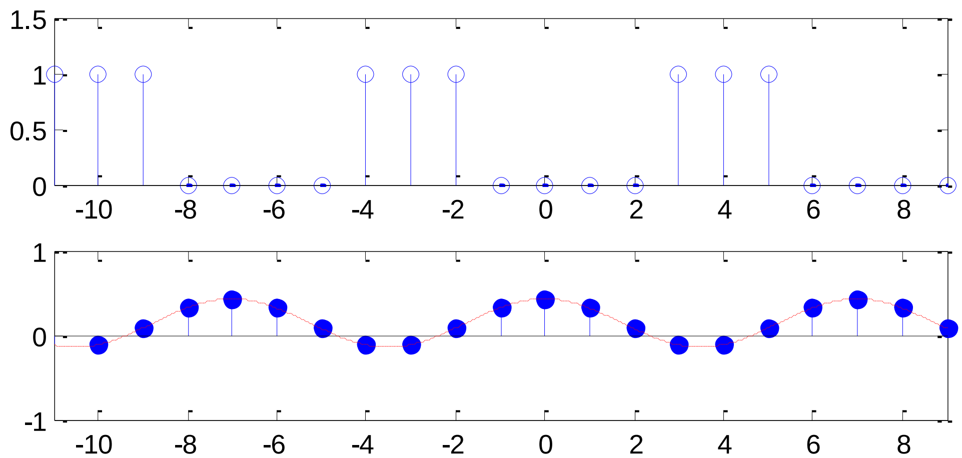


Figure 4-25 The discrete-time periodic signal and its DTFSC - pulse size $K_p = 3, K_0 = 7$

In this case, the pulse size is 3 samples in time domain lasting 3/7 sconds. In the frequency domain the main lobe is approximately 2.5 bins wide, from which we get a bandwidth of 2.4 Hz, which is pretty close to actual the bandwith of 7/3 Hz = 2.3 Hz, the inverse of the pulse duration time. The sinc tails are no longer seen because the pulse size is narrowing and requires more than half the bandwidth and hence we are now getting a form of alaising effect.

We see the same thing in the next graph, where the pulse is very narrow compared to the period and ovelaps the spectrum of the previous pulse.

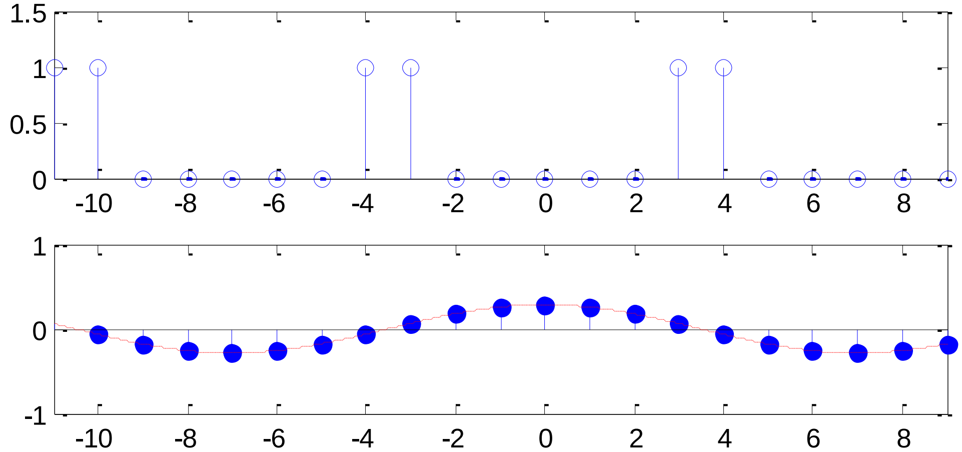


Figure 4.27 The discrete-time periodic signal and its DTFSC - pulse size
$K_p = 2, K_0 = 7$

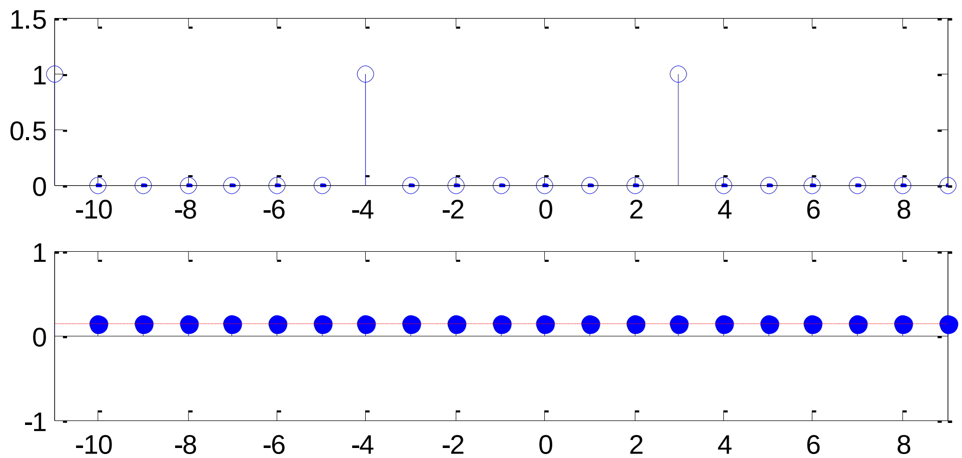


Figure 4.28 The discrete-time periodic signal and its DTFSC - pulse size
$K_p = 1, K_0 = 7$

In this case, we have devolved to case similar to the first one, the pulses are located 7 samples apart and hence in frequency domain they are much closer, which being the inverse of period time. In frequency domain each pulse is one bin apart with resolution of $2\pi / 7$ Hz.

```
% Figure 4.28
clf
K0 = 7;
N = 1;
a = [1];
b = [ a 0 0 0 0 0 0 ];
```

```
c = [ b b b];
subplot(2,1,1)
t = -11: 9;
stem(t, c)
axis([-11 9 0 1.5])
subplot(2,1,2)
n = -14: 13;
xom = (N/K0)*diric(2*n*pi/K0, N);
grid on
stem(n, xom, 'filled')
hold on
n = -11: .01: 9;
xom2 = (N/K0)*diric(2*n*pi/K0, N);
plot(n, xom2, '-.r' )
axis([-11 9 -1 1])
```

Discrete-time Fourier transform of aperiodic signals

Now we will discuss the two most important cases. Both are the discrete-time Fourier transform, for aperiodic and periodic signals.

We start first with the DTFT of a aperiodic signal.

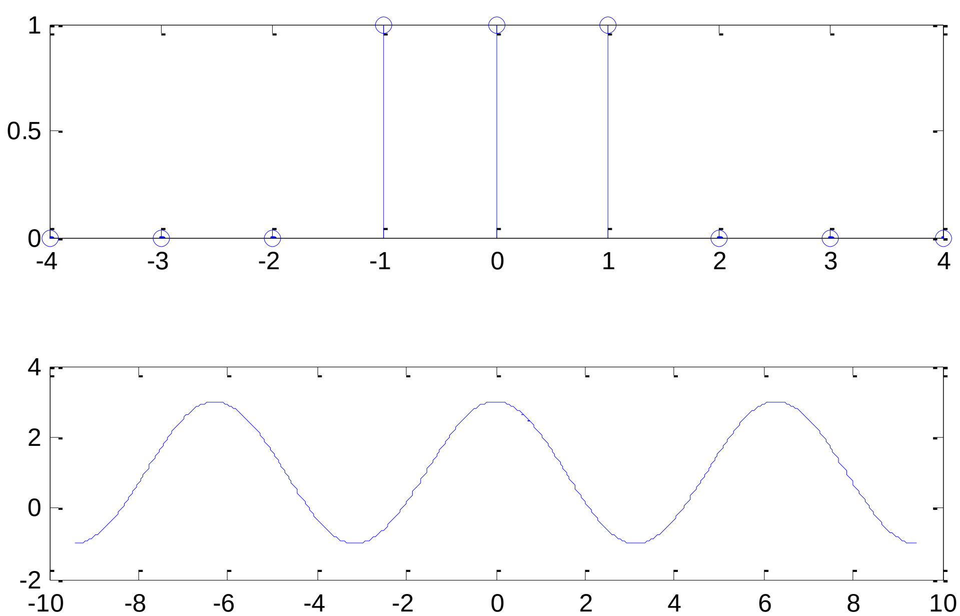


Figure 4.29 The discrete-time aperiodic signal and its DTFSC - pulse size = 3

The spectrum is continuous and repeating.

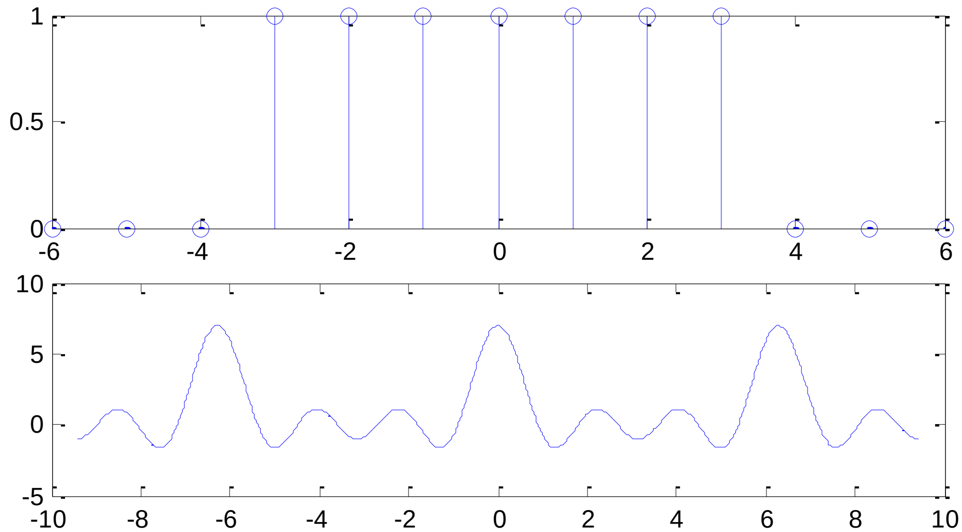


Figure 4.30 - The discrete-time aperiodic signal and its DTFSC - pulse size = 7

This is a DTFT of similar signal with a longer pulse length. Just as we said in the previous case, a longer pulse has smaller frequency content so the spectrum is not aliased and we do see the sinc pattern in between the main lobes.

However, now we ask a key question. If DTFT assumes that the signal has an infinite period, then why does it matter how long the pulse is relative to the number of points shown?

Answer for that can be seen in the limits of the calculation of the DTFT.

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{k=-N}^{N} 1e^{-j\Omega n}$$

$$= \frac{\sin\left(\frac{2N+1}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}$$

The DTFT is being calculated only with the width of the pulse as a variable and nothing else. Since larger the number of discrete harmonics, the better the resolution, this becomes more of a resolution question rather than the issue of the time width of the pulse. We are measuring the width of the pulse by the number of samples and not time. When we are looking at the spectrum of a discrete aperiodic signal, we are seeing only the frequency content in that portion.

In case (a), only three functions have been added to produce the curve, one for each 'impulse sample' of the signal, In case (b), we have used 7 such functions, each with a resolution of $2\pi/N$. Although both are periodic with 2π, the second case has better resolution.

Then we have another question. Why is it not being interpreted as a train of impulses? Why aren't we getting a spectrum same as 4.26? What else is going on here? Actually it is interpreting it as impulses, just that it is adding all the "impulse responses of each of the N impulses" instead of giving the response of just one. So the more of these impulses there are, the more functions are being added to produce the spectrum. If we wanted the signal to be interpreted as independent impulse train, then we would set the pulse period to $K_p = 1$.

DTFT of a periodic signal

Now we will see if we can bring in the idea of a period into DTFT. The DTFT of a periodic signal is the sampled version of its DTFS complex coefficients.

We write this as

$$X(\Omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta\ \Omega - n\Omega_0$$

So this means that the DTFT of a periodic signal is the sampled version of the DTFS. We computed the DTFS of square pulses in Case 3. The DTSFC computed of a periodic discrete waveform already appear to be discrete so what are we doing extra here? Well not much. The process here just multiplies the coefficients by an impulse train in the frequency domain of an assumed fundamental frequency. This assumed frequency can be different than the one used in DTFS or it can be the same. If it is same, we will get the same values as DTFS except scaled by 2π. If not the same, then we will get a decimated version.

All of these ideas about Fourier series, the signal periodicity and the Fourier transform are so closely related that they have tendency to be confused and forgotten. But as we shall see later when looking analysis of real signals using Fourier transform that they tool is very forgiving and we can get useful information even we do not do the "correct" thing, i.e. use the Fourier transform when as Fourier series is the correct answer.

In the next chapter we will talk about DFT and FFT which is how these tools are used.

Summary

1. Fourier series is not intended for aperiodic signals.
2. Fourier transform is an extension of the Fourier series and applies to aperiodic signals by assuming that the period of the signal is infinite.
3. This assumption results in a spectrum that is continuous since the fundamental frequency is now zero.
4. The continuous-time Fourier Transform (CTFT) of aperiodic signals is continuous.
5. The discrete-time Fourier Transform (DTFT) is developed in exactly the same way as the CTFT assuming that fundamental period approaches infinity.
6. This also results in a continuous spectrum but on that repeats same as DTFSC.
7. We can use the DTFSC to compute a DTFT of a periodic signal.
8. The DTFT of a periodic signal is a sampled version of the DTFSC.

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